R'B.A.N.M'S HIGHER SECONDARY SCHOOL
BANGALORE-1.

PREFACE.

THIS Part has been written on the same lines as Part I of this book, and is intended for use in the higher forms of Secondary Schools. The chapters have been so written that a teacher can take up the topics in any order he likes. The articles and examples that may be omitted on a first reading are marked with an asterisk.

We take this opportunity of offering our cordial thanks to Mr. T. K. Venkatarama Aiyar of the Presidency College, Mr. S. Balakrishna Aiyar of the Kumbakonam College and Mr. A. Thiruvenkatatha Aiyangar, Headmaster, Pattamadai High School, for their kind criticisms and suggestions, and for reading through the proofs.

Criticisms and suggestions towards the improvement of the book will be thankfully received.

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V. VENKATASUBBAYYA.

15*th January*, 1914.

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ELEMENTARY MATHEMATICS.

PART II.

CHAPTER I.

ALGEBRAIC MULTIPLICATION AND DIVISION.

§ 1. Multiplication of Negative quantities. In Chapter VI, Part I, we have defined multiplication as repeated addition.

Thus $+ 5 \times (+ 4)$ means $+ 5$ taken 4 times,

i.e., $(+ 5) \times (+ 4) = + 5 + 5 + 5 + 5 = + 20$.

Also $(- 5) \times (+ 4)$ means $(- 5)$ taken 4 times,

i.e., $(- 5) \times (+ 4) = - 5 - 5 - 5 - 5 = - 20$.

But $(+ 5) \times (- 4)$, which, according to our definition of multiplication, stands for "$+ 5$ taken $- 4$ times," is meaningless, and let us see what meaning may be given to it.

Now the identity $a \times b = b \times a$ has been proved to hold when a and b are positive numbers (*vide* p. 106, Part I); and since the letters used in Algebra may stand for any numbers, the meaning that we give to $(+ 5) \times (- 4)$ should not be inconsistent with the identity $a \times b = b \times a$. Then $(+ 5) \times (- 4)$ must be taken to denote the same thing as $(- 4) \times (+ 5)$ which, we have seen, means $(- 4)$ taken 5 times, *i.e.*, $- 4 - 4 - 4 - 4 - 4$ or $- 20$.

The same result is obtained if $(+ 5)$ be multiplied by $(+ 4)$ and the sign of the product changed.

We therefore have, *multiplication by* $(- 4)$ *is the same as* multiplication by $(+ 4)$ and changing the sign of the product.

Retaining the same meaning for multiplying by -4, $(-5) \times (-4)$ would mean multiplying (-5) by $(+4)$ and changing the sign.

i.e. $(-5) \times (-4) = -\{(-5) \times (+4)\}$
$= -(-20) = +20.$

Thus we have

$(+5) \times (+4) = +20$ ... ... (i)
$(-5) \times (+4) = -20$ ... ... (ii)
$(+5) \times (-4) = -20$ ... ... (iii)
$(-5) \times (-4) = +20$ ... ... (iv)

From the results numbered (i) to (iv) we see that the product is positive when the two factors have *like* signs and negative when the two factors have *unlike* signs.

And, in general, we have (remembering that the product of a and b is written ab)

$$\left.\begin{aligned} (+a) \times (+b) &= +ab \\ (-a) \times (+b) &= -ab \\ (+a) \times (-b) &= -ab \\ (-a) \times (-b) &= +ab \end{aligned}\right\} \text{(A)}$$

From (A) we deduce the rule of signs for the multiplication of two quantities, *viz.*, **like signs yield + ; unlike signs yield −.**

Note.—It may be pointed out that -4 means $+4$ with the direction reversed. Thus $(+5) \times (-4)$ would naturally mean $(+5) \times (+4)$ with the direction reversed, *i.e.*, with the sign changed.

§ 2. Graphical illustration of the above rule. $(+5) \times (+4)$ may be graphically represented by the area of a rectangle whose sides are $+5$ and $+4$. Taking, as before, distances measured to the right and upwards as positive and those measured to the left and

downwards as negative, and also regarding the area of a rectangle described in the counter-clockwise direction as positive and that described in the clockwise direction as negative we have the following four figures to represent the four results marked A in the preceding article.

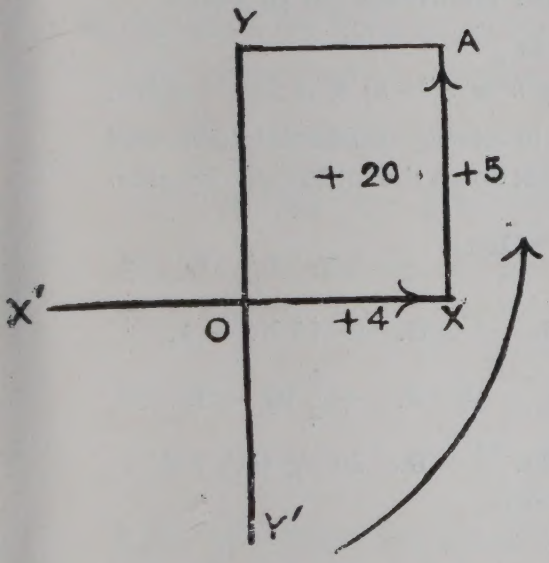

FIG. 1.

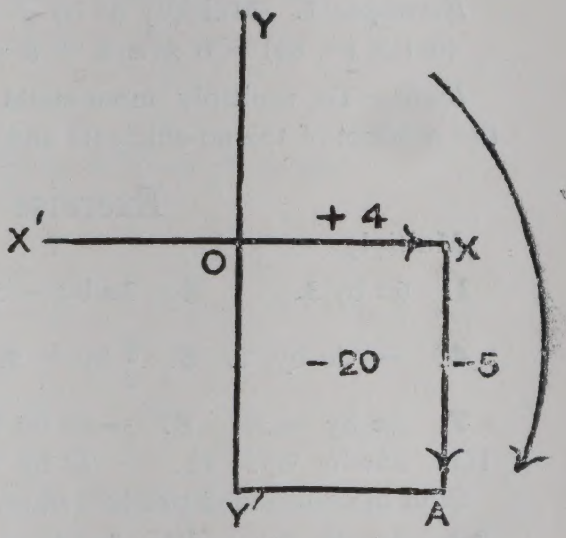

FIG. 2.

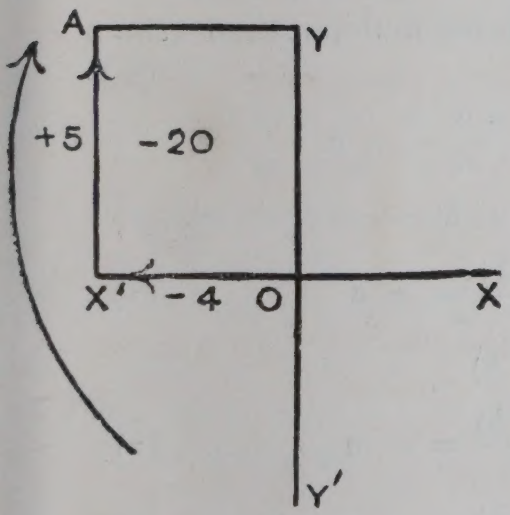

FIG. 3.

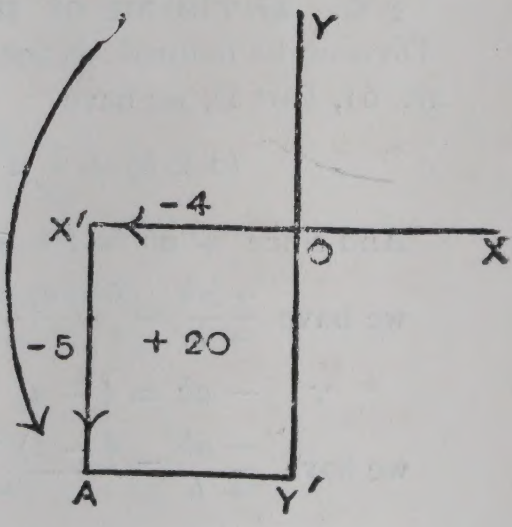

FIG. 4.

§ **3.** Since the result $a \times b = b \times a$ is universally true, *i.e.*, for all values of a and b, hence also, for all values of a, b and c, we have $\mathbf{abc} = (ab)c = (ba)c = \mathbf{bac}$

$$\mathbf{bac} = b(ac) = b(ca) = \mathbf{bca}$$

and so on, *i.e.*, *the factors of a product may be taken in any order.* (*Vide* Art. 62, Part I and Exercises on p. 140).

Example 1. Multiply $6a$ by $-8b$.

$(6a) \times (-8b) = 6 \times a \times -8 \times b = 6(-8) \times a.b = -48ab.$

Rule.—To multiply monomials involving different letters, find the product of the co-efficients and set down the letters side by side.

Exercise—Oral.

Multiply :—

1. $6a$ by 3. **2.** $3a$ by -5. **3.** $-7b$ by -1.

4. $-\cdot5a$ by $\cdot8$. **5.** $\frac{a}{2}$ by -3. **6.** $-\frac{b}{4}$ by -16.

7. $2a$ by $-3b$. **8.** $-6c$ by $8d$. **9.** $3c$ by $4b$.

10. $25x$ by $\cdot3y$. **11.** $-\cdot8l$ by $\cdot9m$.

Find the continued product of :—

12. $3a$, $2b$, $3c$. **13.** $3a$, $4b$, $5c$. **14.** $-3a$, $-6b$, $-8c$.

15. $-\frac{1}{2}a$, $-\frac{1}{3}b$, $-c$.

§ **4. Division of negative quantities.** If Division be defined as the *inverse* of multiplication (*vide* art. 61, Part I), we have

$$(a \times b) \div b = a; \text{ or } \frac{a \times b}{b} = a.$$

And since $+ ab = (+ a) \times (+ b)$

we have $$\frac{+ ab}{+ b} = \frac{(+ a) \times (+ b)}{+ b} = + a \quad \ldots \quad (1)$$

$\because \quad - ab = (- a) \times (+ b)$

we have $$\frac{- ab}{+ b} = \frac{(- a) \times (+ b)}{+ b} = - a \quad \ldots \quad (2)$$

$\because \quad - ab = (+ a) \times (- b)$

we have $\frac{-ab}{-b} = \frac{(+a) \times (-b)}{-b} = +a$ ... (3)

Similarly $\frac{+ab}{-b} = \frac{(-a) \times (-b)}{-b} = -a$ ... (4)

From (1), (2), (3) and (4), we see that the rule of signs is the same for division as for multiplication ; *viz.*,

Like signs yield + ; unlike signs yield —

Ex. (1). Divide $-15a$ by -3.

$$(-15a) \div (-3) = \frac{-15a}{-3} = +\tfrac{15}{3} \cdot a = +5a.$$

Ex. (2). Divide $-6a$ by $3b$.

$$-6a \div 3b = \frac{-6}{3} \cdot \frac{a}{b} = -2\frac{a}{b}.$$

Ex. (3). Divide $18\,ab$ by $-6b$.

$$(18\,ab) \div (-6b) = \frac{18 \cdot a \cdot b}{-6 \cdot b} = -\frac{18}{6} \cdot a \cdot \frac{b}{b} = -3a.$$

Rule.—To divide monomials involving different letters or same powers of a letter, find the quotient of the numerical co-efficients, write down the letters in the numerator and denominator after cancelling like letters.

Exercise—Oral.

Divide—

1. $+36$ by $+3$.
2. $+84$ by -7.
3. -36 by -12.
4. $-3\frac{1}{3}$ by $1\frac{2}{3}$.

Perform the operations indicated below :—

5. $14 \div (-2)$.
6. $(-8\frac{1}{3})$ by $(+1\frac{1}{4})$.
7. $(-36) \div (-4)$
8. $(-\cdot35)$ by $(-\cdot7)$.

Divide—

9. $36a$ by -4.
10. $-44b$ by $-11b$.
11. $-8ab$ by $2a$.
12. $-8abc$ by $-4ab$.
13. $-3\,ax$ by $-3x$.
14. $\frac{-3xy}{5}$ by $-2\frac{1}{2}x$.

15. $-\frac{yb}{15}$ by $-\frac{y}{3}$. 16. $-10\,abc$ by $-5ab$.

Divide the product of :—

17. $-3x$ and $10y$ by 15. 18. $3a$, $-4b$, $-5c$ by $12ab$.

§ 5. Multiplication of monomials.

$(-x)^2 = (-x) \times (-x) = +(x^2) = +x^2$

$(-x)^3 = (-x) \times (-x) \times (-x)$

$= +x^2 \times -x = -x^3$

Continuing the process it is seen $(-x)^4 = +x^4$,

$(-x)^5 = -x^5$ &c., and in general,

$(-x)^n = +x^n$ if n is **even**

$(-x)^n = -x^n$ if n is **odd**, or in other words, **an even power of a negative quantity is positive whilst an odd power is negative.**

In Art. 54, Part I, it has been proved that

$$a^m \times a^n = a^{m+n} \ldots\ldots\ldots\

Ex. 2. Multiply $-6x^2y^3z$ by $-8xy^3z^2ab$

$$(-6x^2y^3z) \times (-8xy^3z^2ab)$$
$$= -6 \cdot x^2 \cdot y^3 \cdot z \cdot -8 \cdot x \cdot y^3 \cdot z^2 \cdot a \cdot b.$$
$$= -6 \cdot -8 \cdot x^2 \cdot x \cdot y^3 \cdot y^3 \cdot z \cdot z^2 \cdot a \cdot b$$
$$= 48\,x^3y^6z^3ab.$$

Rule.—To multiply two or more monomials together, multiply the numerical co-efficients together, then add the indices of like letters and set down side by side the unlike letters with their indices.

Ex. 3. Find the value of—

$3a^2bc^3d$, when $a = -1$, $b = -2$, $c = -3$, $d = -4$.

$$3a^2bc^3d = 3.(-1)^2.(-2)\,(-3)^3.(-4)$$
$$= 3 \cdot +1 \cdot -2 \cdot -27 \cdot -4$$
$$= 3 \cdot -2 \cdot +108 = -648.$$

Ex. 4. Find the square of $3a^4b^3c^2$

$$(3a^4b^3c^2)^2 = 3^2 \cdot (a^4)^2 \cdot (b^3)^2 \cdot (c^2)^2$$
$$= 9 \cdot a^8 \cdot b^6 \cdot c^4 = 9a^8b^6c^4.$$

Exercise I (a).

Simplify—

1. $(-a)\,(-b)\,a^2$. **2.** $(-9x) \cdot (4x^3)$. **3.** $(8a^2b) \cdot (7b^2c)$.

4. $(-6a^2bc)\,(abc)^2$.

Find the product of—

5. $5a^4b^2c$ and $-63a^3b^2cd^2$.

6. $\frac{a}{8}bc^3$ and $-\frac{4}{5}a^3c^2d$.

7. $-\frac{1}{6}l^3m^2n$ and $-\frac{7}{6}m^3n^2pq$.

Find the squares of—

8. $-8a^3b^2c$. **9.** $\frac{5}{6}b^3cd^2z$.

Find the cubes of—

10. $-8ab^3c$. **11.** $\frac{1}{9}b^3de$. **12.** $125a^3c$.

Find the continued product of—

13. $-8a^2b$, $-3b^3c^2d$, $8a^2cxy$. **14.** x, $2x^2$, $5x^3$ and by^2.

Find the value of—

(1) $(a + b)^2$ when $a = -8$, $b = 6$.

(2) $7a^3b^2c + 6c^2d$ when $a = -2$, $b = -3$, $c = 0$, $d = -6$.

(3) $(a + b + c - d)^2 \ (b + c + d)^3$ when $a = 0$, $b = -1$, $c = 3$, $d = -5$.

(4) $(-8a^3b^2c)^p$ when $a = 1$, $b = 4$, $c = 3$, $p = 2$.

§ **6. Division of monomials.** We know that division is the inverse of multiplication. Hence since $a^3 \times a^4 = a^{3+4} = a^7$, it follows that $a^7 \div a^4 = a^3$ and $a^7 \div a^3 = a^4$

or thus $a^7 = a \times a \times a \times a \times a \times a \times a$

and $a^4 = a \times a \times a \times a$.

$$\therefore \quad a^7 \div a^4 = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a}$$

$$= a \times a \times a = a^3.$$

Similarly it may be shown $a^7 \div a^3 = a^4$.

This may be generalised thus :

$$\frac{a^m}{a^n} = a^{m-n} \text{ (when } m > n.)$$

Also $a^4 \div a^7 = \dfrac{a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a}$

$$= \frac{1}{a \times a \times a} = \frac{1}{a} = \frac{1}{a^{7-4}}$$

and, generally, it may be shown that

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ when } m < n.$$

Note.—If we grant that the formula $\frac{a^m}{a^n} = a^{m-n}$ is true not only when $m > n$ but also when m is $< n$, it follows that $\frac{a^4}{a^7} = a^{4-7} = a^{-3}$; but it has been shown $\frac{a}{a^7} = \frac{1}{a^3}$. $\therefore$ a^{-3} may be regarded as equivalent to $\frac{1}{a^3}$. Generally $a^{-n} = \frac{1}{a^n}$.

Also $\frac{a^m}{a^m} = a^{m-m} = a^0$. But $\frac{a^m}{a^m} = 1$.

(Any quantity divided by the same quantity gives us 1).

$$\therefore \quad a^0 = 1.$$

Ex. 1. Divide $16a^4$ by $2a$.

$$\frac{16a^4}{2a} = \frac{16}{2} \cdot a^{4-1} = 8a^3.$$

Ex. 2. Divide $-36a^5b^3c^2$ by $-9ab^2$.

$$\frac{-36a^5b^3c^2}{-9ab^2} = \frac{36}{9} \cdot \frac{a^5}{a} \cdot \frac{b^3}{b^2} \cdot c^2 = 4a^4bc^2.$$

Ex. 3. Simplify the fraction—

$$\frac{16a^2b^3d^2}{24a^5bc^2d^3} = \frac{2 \times 8}{3 \times 8} \times \frac{a^2}{a^5} \cdot \frac{b^3}{b} \cdot \frac{d^2}{d^3} \cdot \frac{1}{c^2}$$

$$= \frac{2}{3} \times \frac{1}{a^3} \cdot b^2 \cdot \frac{1}{d} \cdot \frac{1}{c^2} = \frac{2}{3} \frac{b^2}{a^3dc^2}.$$

Exercise I (b).

Divide—

1. a^7 by a^3. **2.** $-7x^2y$ by xy. **3.** x^2yz by xz.
4. $-49a^3b^2c^3$ by $7a^2bc$. **5.** $(-a)^5 \div (-a)^4$.
6. $9x^m \div 3x^p$. **7.** $9a^2b^2c^2 \div (-2ab)^2$.
8. $-32x^my^n$ by $4x^p \cdot y^q$. **9.** $a^{2m}b^{3n}c^{4p}$ by $a^mb^{2n}c^{2p}$.
10. $-8l^3m^3n^3$ by $4l^2m^2n^2$.

Find the value of—

11. $\frac{8p^2q^2r^2s}{3(pq)^2r^3}$ when $p = 3$, $q = 1$, $r = 2$ and $s = 4$.

12. $\frac{(abcd)^3}{a^2b^2c^2}$. **13.** $\frac{3a}{5b} \times \frac{6c}{7a}$. **14.** $\frac{3x^3}{7y^2z^2} \div \frac{7yz}{8x}$.

15. $\frac{2a^2}{3bc} \times \frac{6b^2}{4ac} \times \frac{9c^2}{7ab}$. **16.** $\frac{8x^2}{y^2} \times \frac{9y^2}{x^2} \div \frac{xy}{z^2}$.

§ 7. Multiplication of an expression by a monomial. We have seen (*vide* page 108, Part I) that $(b + c)\,a = ab + ac$ when a, b, c are positive numbers and that $(b - c)\,a = ab - ac$ when $b > c$.

We can easily show that these identities are true for all values of a, b and c.

To show that $(b - c)\, a = ab - ac$ when $b < c$, and a negative.

Let $a = -x$, so that x is positive, then $(b - c)\, a$

$= \{ -(c - b) \} (-x)$

$= x(c - b)$ [rule of signs]

$= xc - xb$ since $c > b$

$= (-x) \cdot (-c) + (-x) \cdot b$ (rule of signs)

$= a(-c) + ab = -ac + ab$

$= ab - ac.$

These may be extended and we have in general,

$k(a - b + c - d \ldots) = ka - kb + kc - kd \ldots\ldots$

Ex. 1. Multiply:— $(8a - 9b + 3c)$ by $4d$.

$(8a - 9b + 3c)4d = 8a.4d - 9b.4d + 3c.4d = 32ad - 36bd + 12cd.$

Ex. 2. $(8a^2 - 9ab + 6b^2)$ by $-4c^2$.

$(8a^2 - 9ab + 6b^2) \times (-4c^2) = 8a^2 \times (-4c^2) - 9ab \times (-4c^2)$

$+ 6b^2 (-4c^2) = 32a^2c^2 + 36\, abc^2 - 24\, b^2c^2.$

N.B.— In actual practice the prodnct may at once be written down.

Exercise I (c).

Find the product of—

1. $(a + 5b)$ and $6b$. **2.** $(7a + 3b - 2c)$ and $8c$.

3. $x^3 - 3x^2 + 4x - 1$ and $6x$.

4. $a^2 - 9ab + 6b^2$ and $4a^3$. **5.** $2ab + 3ac - 5bc$ and $6a^2$.

6. $a^3 - 3a^2 b + 6a^2 - 4b$ and $-7a^2b$.

Simplify—

7. $(a - 3b)\, 4c + (6b - 7c)\, 9c$.

8. $5 - x \{ 4 - 3(x - 2) \}$.

9. $x(y - z) + y(x - z) + z(x - y)$.

10. $5x^3 - 4x^2 (2x + 1) - 3x (2x + 5)$.

§ 8. Division of an expression by a monomial. Applying the rule of signs and the definition

of division as the inverse of multiplication, we can easily show that $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ and $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ for all values of a, b and c and more generally

$$\frac{a-b+c-d\ldots}{k} = \frac{a}{k} - \frac{b}{k} + \frac{c}{k} - \frac{d}{k}\ldots$$

Ex. 1. Divide $3a^3 - 6a^2b + 9ab^2 -$ by $3a$.

$$\frac{3a^3-6a^2b+9ab^2}{3a} = \frac{3a^3}{3a} - \frac{6a^2b}{3a} + \frac{9ab^2}{3a} = a^2 - 2ab + 3b^2.$$

Ex. 2. Simplify $-\frac{2a^2+6ab}{a} - \frac{3b^2-5ab}{b}$

The expression $= (2a+3b)-(3b-5a) = 2a+3b-3b+5a = 7a$.

Exercise I (d).

Divide—

1. $12-16x+20x^2$ by 4.
2. $a^3-3a^2b+3ab^2+$ by a.
3. $8a^2b-12a^3bc+20a^2bc$ by $4ab$.
4. $5a^8b^8-15a^7b^9$ by $5a^6b^5$.
5. $7b^5a^8-21a^8b^6$ by a^7b^5.
6. $9x^5y^9-y^{12}x^2$ by y^9x^2.
7. $x^2y^2+xy^2z+x^2yz$ by xy.
8. $3a^2bc-6ab^2c+12abc^2$ by $3abc$.

Simplify by performing the operations indicated—

9. $5c-\frac{8ab+9ac}{a}$.
10. $\frac{a^2x+acx+ax^2}{ax} - \frac{b^2x+bcx+bx^2}{bx}$.

§ 9. Multiplication of two or more expressions.

Ex. 1. Multiply $(a+b+c)$ by $(x+y)$.

$(a+b+c)(x+y) = (a+b+c)x + (a+b+c)y = ax+bx+cx+ay+by+cy$.

Ex. 2. Find the product of $(x+3)$ and $(x+5)$.

$(x+3)(x+5) = (x+3)x + (x+3)5 = x^2+3x+5x+15 = x^2+8x+15$.

Ex. 3. Multiply x^2+2x+3 by $x-1$.

$(x^2+2x+3)(x-1) = (x^2+2x+3)x - (x^2+2x+3)$.

The same may be represented thus:

$$\begin{array}{r} x^2+2x+3 \\ x-1 \\ \hline x^3+2x^2+3x \\ \quad -x^2-2x-3 \\ \hline x^3+x^2+x-3. \end{array}$$

Hence the following rule:—"Arrange each expression in descending powers of some letter. Write down the two expressions as shown, one under the other, multiply each term of the first expression by each term of the second, beginning at the left hand side, taking care to see that like terms are placed in the same vertical column finally add the terms together to obtain the product."

The process of long multiplication explained in Chapter VI, Part I is only a particular case of this.

Exercise I (e).

Multiply—

1. $a^4-a^3b+a^2b^2-ab^3+b^4$ by $a+b$.

2. $a^2-a^2b^2+b^4$ by a^2+b^2. **3.** $2x^2+x-5$ by $x-3$.

4. $4x^2+7x-8$ by $9x+2$. **5.** a^2-ab+b^2 by a^2+ab+b^2.

6. $(3a+2b+9c)$ by $3a-2b-9c$.

7. $2x^3-3x^2y+3xy^2-y^3$ by $2x^3+3x^2y-3xy^2-y^3$.

8. $x-y+2$ by $x+y-1$.

Find the square of

9. $2x-5$. **10.** $2a+3b$.

11. $3a+4b-9c$. **12.** $4c-3b+2d-a$.

§ 10. Division of an expression by another.

Ex. (1). Divide $x^2+5xy+6y^2$ by $x+3y$.

$$\begin{array}{l} x+3y)x^2+5xy+6y^2(x+2y \\ \quad \underline{x^2+3xy} \\ \quad\quad 2xy+6y^2 \\ \quad\quad \underline{2xy+6y^2} \end{array}$$

When the dividend and divisor both consist of several terms, we arrange the dividend and the divisor according to the powers of the same letter beginning with the highest. Divide the first term of the divisor into the dividend. Thus x divided into x^3 gives x; write this quantity on the right hand side as shown and put the term x^2 under the first term of the dividend. In a similar manner by multiplying the remaining term, *viz.*, $3y$ by x and subtracting we obtain $2xy$. Now bring down the next term $6y^2$ and proceed as before. The process very much resembles division in Arithmetic and no further explanation is necessary.

Ex. (2). Divide $1+x^2+x^4$ by $1-x+x^2$.

$$
\begin{array}{l}
1-x+x^2)1+x^2+x^4(1+x+x^2 \\
\quad \underline{1-x+x^2} \\
\qquad x+x^4 \\
\qquad \underline{x-x^2+x^3} \\
\qquad\quad x^2-x^3+x^4 \\
\qquad\quad \underline{x^2-x^3+x^4}
\end{array}
$$

N.B.—Expressions may be arranged in ascending powers also before division is commenced.

Exercise I (f).

Divide and give the remainder, if any, in—

1. x^3+1 by $x+1$. **2.** x^3-1 by $x-1$.

3. x^5+a^5 by $x+a$. **4.** $x^2+7xy+12y^2$ by $x+4y$.

5. $a^3+b^3+c^3-3abc$ by $a+b+c$. **6.** x^6-1 by x^4+x^2+1.

7. $x^3+6x^2+12x+8$ by x^2+4x+4.

8. $x^4+6x^3-8x^2+9x-36$ by $x-2x+5$.

Express as a mixed number—

9. $\dfrac{x^4+6x^3-9x^2+12x+1}{x^2-3x+5}$. **10.** $\dfrac{a^4+2a^3+5}{a^2-1}$.

Simplify—

11. $\dfrac{(x^2+8ax+15a^2)\,(x+4a)}{x^2+7ax+12a^2}$.

12. $\dfrac{(x^3-1)(x^3+x+1)}{x^3-1}$.

Exercise I (g).

1. Remove brackets and then find the value of the expression when $a=1$ and $b=2$

$$4a^2b-[3a^2b-\{a^2b+(ab^2-7ab^2)\}].$$

2. From $(3x+6y)(8x-9y)$ take away the sum of $(7x-2y)\times(9x+3y)$ and $(11x-13y)(x-y)$.

3. Prove the following by division—

(*a*) $\dfrac{x^2+8x+16}{x+3}\equiv x+5+\dfrac{1}{x+3}$ and verify when $x=2$.

(*b*) $\dfrac{x^2-13x+36}{x-7}\equiv x-6-\dfrac{6}{x-7}$ and verify when $x=1$.

4. What must be added to $9x^2+12x+5$ to make it exactly divisible by $3x+2$.

5. Divide ax^3+bx^2+bx+a by $x+1$.

6. Verify the following identities by substituting 1 for a, -1 for b, and -3 for c :—

(i) $a^3+b^3\equiv(a+b)(a^2-ab+b^2)$

(ii) $(a+b)(a+c)\equiv a^2+a(b+c)+bc$

(iii) $a^3+b^3+c^3-3abc$

$\equiv(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.

7. If $a=-1$, $b=-2$, $c=-3$, $d=-4$, find the value of—

(*a*) $(ab+ac+bc)^2$

(*b*) $a^2b^2+a^2c^2+b^2c^2$

(*c*) $a^4+b^4+c^4$

(*d*) $a^3+b^3+c^3$

(*e*) $x^4-6x^3+5x^2-8x+4$ when $x=-2$, $x=-3$, $x=-10$

(*f*) $6x^2+3x+8$ when $x=-3, 0, 4\frac{1}{2}$.

8. Find the value of

$(+2)^n+(-2)^n$ when $n=3$ and $n=4$.

9. Find the continued product of—

(*a*) $(x+1)$, $(x+2)$ and $(x+3)$

(*b*) $(x-y)$, $(y-z)$ and $(x-z)$

(*c*) $n-1$, $n+1$, $2n-1$ and $2n+1$.

10. Pick out the term involving x^3 in the product of—
$(x^3-7x^2+8x-15)(x^2-7x+6)$.

11. The four sides of a quadrilateral figure are respectively $(7a^3-5a^2+3)$ ft., $(8a^2-9a+4)$ ft., $(-9a^2-7a+6)$ft. and $(a^3-11a^2+5a-10)$ ft. Find the perimeter of the figure and the side of a square having the same perimeter as the quadrilateral and check your answers by putting $a=2$.

12. A man makes a profit of £$(4x^3+x^2-x+5)$ by selling a horse for £$(9x^3-3x^2+5)$. What is the cost price of the horse? Express the profit as a fraction of the cost price.

13. A rectangle is $(2x^3-7x^2+4)$ yds. long and $(3x^2-6x+9)$ ft. broad; find its area in square yards, and the cost of matting the rectangular ground in rupees at y pies per square foot.

14. A man walks $(8x^3+2x^2-27x-18)$ miles in $(2x^2-x-6)$ hours. What is his rate, and find the value of x if this rate is equal to 4 miles per hour?

15. Find the area of a triangle whose base measures $(a+b)$ units of length and whose altitude measures $(p+q)$ units and show that the triangle is equal to the sum of four triangles whose bases measure a, p, q, b units respectively and whose altitudes are p, b, a, q, units respectively.

16. The parallel sides of a trapezium are a and b units of length and the distance between the parallel sides is p units. Show that the area of the trapezium is $\frac{1}{2}(a+b)p$. units of area, the unit of area being the square whose side is the unit of length. Using this formula find the area of a trapezium whose parallel sides and the distance between them are (a^2-a+1) ft., (a^2+a+1) ft. and (a^2-1) ft.

17. Examine if the following lengths can be the sides of a right-angled triangle: a^2+4a+3, $2(a+2)$, a^2+4a+5.

18. The area of a triangle the co-ordinates of whose angular points are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the formula $\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$. From this formula

find the area of a triangle whose vertices are $(3, -4)$, $(-5, 6)$, $(-7, -8)$.

19. If a stone is thrown vertically upwards with a velocity of u ft. per second, and if after t seconds its velocity upwards is v ft. per second, it can be shown that $v = u - gt$ where $g = 32$. Find v when (i) $u = 80$, $t = 2$ (ii) $u = 140$, $t = 4$ (iii) $u = 160$ and $t = 5$.

20. If in the previous question the space through which the body moves in t seconds is s ft. it can be shown that $s = ut - \frac{1}{2}gt^2$. Find s when (1) $u = 40$, $t = 3$ and (2) $u = 64$, $t = 4$.

21. A builder sells a house worth Rs. h to an agent at a loss of l per cent. If the agent disposes of it at a gain of p per cent., what does the purchaser pay?

22. The length and breadth of a rectangular room are a ft. and b ft. The walls are c inches thick. A verandah of width v ft. runs all round the room. Find the area of the room, and also the number of slabs (each s in. by d in.) required for paving the verandah.

23. Find the number of square feet in the shaded and unshaded areas represented in the adjoining diagram on the supposition that it is drawn on a scale of 1 inch to a furlong.

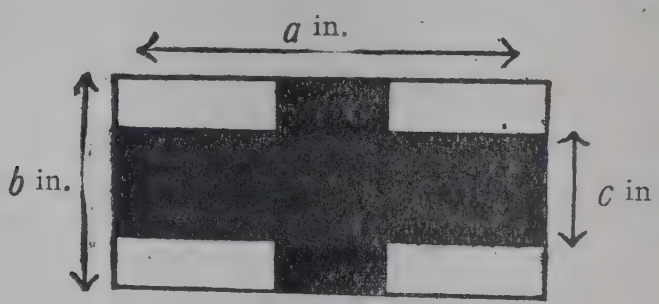

FIG. 5.

24. A pattern made from wood which weighs 750 grams per cubic decimetre weighs 37·5 kilogrammes. A gun metal casting of the same shape and size weighs 336 kilogrammes, the gun metal weighing 8·4 kg. per cubic decimetre. Show that there must be a cavity in the casting. Find what weight of metal would fill it.

Taking the weight of the wood as w grams per cubic decimetre, of the gun metal as g kilograms per cubic decimetre, of the pattern as p kilograms, and of the casting as c kilograms, give an expression for the weight of metal that would fill the cavity.

25. A rectangular subway is to be made w ft. wide and h ft. high. The earth weighs e pounds per cubic foot. How many tons of earth will be removed in making l ft. of the subway?

If air is driven in at one end of the subway at the rate of a cubic feet per minute what will be the speed of the air along the subway?

Give numerical answers for the case in which $w=10$, $h=9$, $e=80$, $l=20$, $a=3000$.

26. An air-pump with a cylinder of volume v is used to exhaust a vessel of volume V. After n strokes of the pump the pressure of the air in the vessel is $\left(\frac{V}{v+V}\right)^n \times p$ where p is the atmospheric pressure. Taking p as 30 inches of mercury, v as 30 cubic inches and V 170 cubic inches, find the pressure in inches of mercury after (1) 5 strokes, (2) 10 strokes.

27. A water main is m ft. in diameter. How many gallons are contained in l miles of it? At how many miles per hour must the water flow out to supply n inhabitants with g gallons per head in 12 hours? [Take 1 cubic foot = 6·2 gallons and the area of a circle = ·8 times the square of the diameter.]

28. (*a*) Divide 1 by $1-x$ and find the quotient to 4 terms. Use this quotient to find the value in decimals of $.\frac{1}{99}$.

(*b*) Divide 1 by $1+x$ and find the quotient to 4 terms. Use this result to find the value in decimals of $\frac{1}{1{\cdot}01}$.

In both the above cases what difference does it make if the value is ascertained from the first two terms alone?

29. When a photograph is to be taken, the plate on which the picture is formed must be placed at a distance u from the lens, this distance being connected with the distance v of the object from the lens by the relation $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$ where f is a length depending or the lens used. If $f=3$ inches, find the distance of the plate from the lens when the distance of the object is (1) 5 ft., (2) 20 ft.

30. At sea level water boils at 212° Fahrenheit. At a height H feet above sea level the boiling point is lowered B degrees below 212°, where B and H are connected by the approximate formula $H = 520\,B + B^2$. If water boils at 182°F at the top of a certain mountain, find its height.

31. If the same formula as in Example 30 holds for a depth of H ft. below sea level where the boiling point is raised B degrees above 212°, what must be the depth of a mine at the bottom of which water boils at 222°F ?

32. A river basin of b square miles receives a rainfall of r inches per 24 hours, half of which drains into the river whose section where it leaves the district is d square yards. Find a formula for the velocity of the river at this point (in feet per second) supposing the whole process to be steady and the speed of the water in the river to be the same at all depths. Give a numerical value to your answer when $b = 160$, $r = \cdot 3$ $d = 200$.

CHAPTER II.

SIMILAR TRIANGLES: SCALE.

§ **11.** When two triangles have three angles of the one respectively equal to the three angles of the other, the triangles are said to be **equiangular** and the sides opposite to the equal angles are said to be **corresponding sides**. Even if the triangles have *two* angles of the one respectively equal to *two* angles of the other, they are *equiangular*, for then the third angle of the one must be equal to the third angle of the other, since the three angles of one triangle are collectively equal to the three angles of any other triangle.

In the figure for graphical illustration of proportion given on page 414 of Part I, you have a number of right-angled triangles such as ON_1P_1, ON_2P_2, equiangular to one another; for $\angle ON_1P_1 = \angle ON_2P_2$ each being a right-angle, and $\angle OP_1N_1 = \angle OP_2N_2$ because OP_1 falls on the parallels P_1N_1, P_2N_2 and makes the corresponding angles equal (*vide* p. 227, Part I). It has been established that

$$\frac{P_1N_1}{ON_1} = \frac{P_2N_2}{ON_2};$$

by measuring OP_1, OP_2 it may also be shown that

$$\frac{P_1N_1}{OP_1} = \frac{P_2N_2}{OP_2} \text{ and } \frac{ON_1}{OP_1} = \frac{ON_2}{OP_2};$$

which shows that the sides about the equal angles are proportional.

Exp. Construct two triangles ABC and DEF, on bases BC, EF 3 cm. and 6 cm. long having the base angles equal to 30° and

50°. Measure the sides of both the triangles and compare.

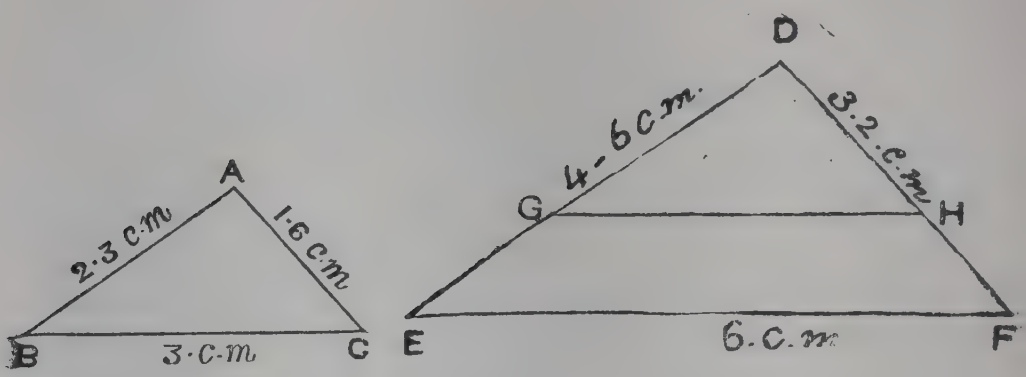

FIG. 6.

By measurement we find

$$AB = 2{\cdot}3 \text{ cm.}; \quad AC = 1{\cdot}6 \text{ cm.}$$

$$DE = 4{\cdot}6 \text{ cm.}; \quad DF = 3{\cdot}2 \text{ cm.}$$

and it is given that BC = 3 cm. and EF = 6 cm.

$$\therefore \frac{BC}{EF} = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \frac{AB}{DE} = \frac{2{\cdot}3}{4{\cdot}6} = \frac{1}{2}$$

$$\therefore \frac{AC}{DF} = \frac{1{\cdot}6}{3{\cdot}2} = \frac{1}{2}$$

$$\therefore \frac{BC}{EF} = \frac{AB}{DE} = \frac{AC}{DF} \quad \ldots \quad \ldots \quad \ldots \quad 1)$$

Repeat the experiment with different values to BC, EF and the base angles.

In all these cases the equality of the ratios in (1) will be found to hold.

Hence we learn that **if two triangles are equiangular the corresponding sides are proportional.** ... ... (A)

Def.—Two rectilineal figures which have the angles of the one respectively equal to the angles of the other and which have the sides about the equal angles proportional are said to be **similar.** The proposition given above may

therefore be worded thus :—*if two triangles are equiangular they are also similar.*

It may be formally proved thus :

* *Lemma.*—**A straight line drawn parallel to the base of a △ cuts the sides proportionally.**

Let ABC be a △ and DE be drawn parallel to BC cutting AB, AC at D and E. Then AD : DB shall be equal to AE : EC.

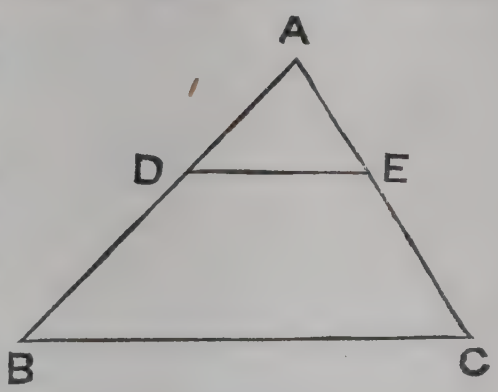

FIG. 7.

Suppose D divides AB in the ratio $m : n$, so that, if AD be divided into m equal parts, then DB will contain n such equal parts.

If, from the points of division in AD, DB, parallels be drawn to BC, these parallels will divide the segments AE, EC, into parts which are all equal. (*Vide* page 329. Part I). And of these equal parts AE will contain m, and EC will contain n

Hence AE : EC = $m : n$

and AD : DB = $m : n$

$\therefore$ AD : DB = AE : EC.

The student is recommended to draw different figures and verify this lemma by measurement.

Now to prove the proposition (A) given on page 20.

Let ABC, DEF be two equiangular triangles having the angles A, B, C respectively equal to the angles D, E, F.

From DE, DF, cut off DG, DH respectively equal to AB, AC.

Then, $\because$ AB = DG

AC = DH

$\angle A = \angle D$

$\therefore \triangle ABC = \triangle DGH$

$$\therefore \angle DGH = \angle ABC = \angle DEF.$$

$$\therefore GH \text{ is parallel to } EF.$$

$$\therefore \frac{GE}{DG} = \frac{HF}{DH}.$$

Add *unity* to both sides

$$\frac{GE + DG}{DG} = \frac{HF + DH}{DH}$$

$$\therefore \frac{DE}{DG} = \frac{DF}{DH}$$

$$i.e. \quad \frac{DE}{AB} = \frac{DF}{AC}.$$

Similarly it may be shown that $\frac{DE}{AB} = \frac{EF}{BC}$.

The proposition (A) may be more distinctly worded thus :

If two triangles are equiangular to one another the sides about the equal angles are proportional, one pair of corresponding sides being the antecedents, and the other pair the consequents.

Exercise II (a).

1. Construct a triangle on a base 4·5 c m. long, with the base angles equal to 40° and 50°. Find the lengths of the other sides. If a triangle of a similar shape is constructed on a base 6·7 cm. long, calculate the lengths of the other sides and check your result by measurement.

2. There are two equiangular triangles, a pair of whose corresponding sides are 114·3 yds. and 129 yds. If the perimeter of the first triangle is half a mile, what is the perimeter of the second ?

3. When the altitude of the sun is 60°, a tree whose height is 30 ft. casts a shadow 17·3 ft. Calculate the length of the shadow cast at the same time by a lamp post 6 ft. high and check your result by drawing a neat figure.

4. ABC is a triangle whose sides a, b, c are 5·3, 6·4, 3·9 cm. respectively. In BC a point D is taken such that BD = 3 cm. Through D a straight line DE is drawn parallel to BA meeting CA in E. Calculate the lengths of the sides of the △ CDE and verify by measurement.

5. Describe a triangle with sides 10·8 cm., 8·9 cm., 7·8 cm. long; and another △ with sides 21·6, 17·8, 15·6 cm. long. What inference can you draw regarding the shape of the triangles. Test your inference by measurement.

6. Repeat exercise 5 with the sides of the second triangle, halves of the sides of the first.

7. In the adjoining figure show that $\frac{AB}{BE}=\frac{DC}{CF}$ by actual measurement and give a formal proof by drawing a line through D parallel to ABE or by joining DE cutting BC at G.

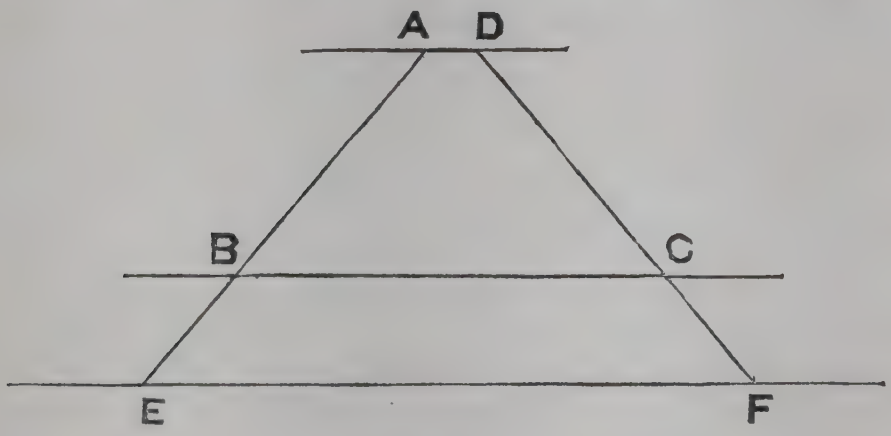

FIG. 8.

8. Figure 9 shows the vertical section of a tiled roof resting on four pillars as shown in the figure. There is symmetry about the dotted line. The distances from the dotted line of the two pillars on one of it, are 8 ft. and 12 ft. respectively, and the pillars are 9 ft. and 6 ft. high. Calculate the distances of the tops of the pillars from the highest point of the roof in the section.

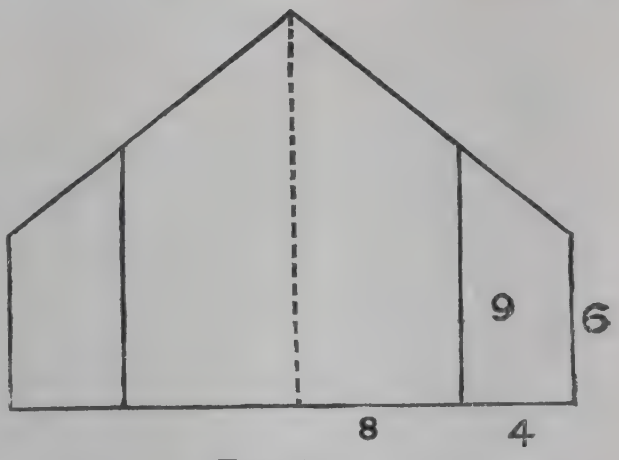

FIG. 9.

*§**12**. Draw with the help of your protractor an angle XOY (say $30\frac{0}{2}$). In one of the arms of the angle, say OY, take any point P. From P draw P M $\perp$ to O X. Measure and find the ratio $\frac{PM}{OP}$. Take any other point P′ in OY and draw P′M, $\perp$ to OX. Find also $\frac{P'M'}{OP'}$. See if these ratios are equal. Account for their equality.

The ratio $\frac{PM}{OP}$, *i.e.*, the ratio of the side opposite the angle XOY to the hypotenuse is called the **sine** of the angle, and does not change as long as the angle is the same.

In the same figure, show that $\frac{OM}{OP} = \frac{OM'}{OP'}$.

The ratio $\frac{OM}{OP}$, *i.e.*, the ratio of the side adjacent to the angle XOY to the hypotenuse is called the **cosine** of the angle and does not vary so long as the angle is the same.

In the same figure, find by measurement, $\frac{PM}{OM}$, $\frac{P'M'}{OM'}$ and see if they are equal. Account for their equality.

The ratio $\frac{PM}{OM}$, *i.e.*, the ratio of the side opposite the angle XOY to the side adjacent to it is called the **tangent** of the angle.

Draw an angle of 60°, find the value of its sine, cosine and tangent.

§ 13. Construction of Scale. With the help of our scale we have been hitherto measuring lengths in either inches and tenths of an inch or in centimetres and millimetres. We have also seen in the chapter on decimals that 2·75 inches means 2 inches, 7 tenths of an inch and 5 hundredths of an inch, though we have not indicated there how to measure lengths to hundredths of an inch. We shall now proceed to construct a scale for ourselves by means of which we can measure lengths to

hundredths of an inch. It is highly important that we should be able to construct our own scale when we have to draw plans to a given scale even to read off inches and tenths of an inch.

Draw a line AB 5 in. long. Divide it into 5 equal parts according to the construction given on page 329. (It is not necessary to show more than the beginning and the end of the 5 parallels.) To show tenths of an inch divide one of

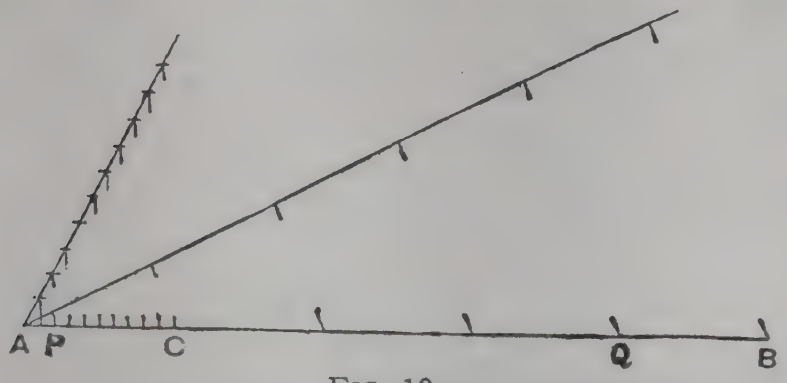

FIG. 10.

the end inches, say AO, into 10 equal parts according to the same construction. Now you can measure lengths to any number of inches and tenths up to 5 inches. For instance, the length PQ = 3·8 inches.

Note the difference between the scale you have constructed and your wooden scale. This scale is much simpler and less complicated in marks and divisions than the instrument you use. In measuring lengths, say 3·6 and 4·7 inches, with your ruler, you start from the reading *zero* and measure in the same direction 3·6 and 4·7 inches, which method requires the division of each inch into ten equal parts. But, in measuring the same lengths, 3·6 and 4·7 inches, with the constructed scale, you start from the reading zero and measure in opposite directions, in one direction for the integral number of inches and in the other

for decimals of an inch. This system requires only one division to be divided into ten equal parts.

Exercise II (b).

1. Draw from your own scalelines of the following lengths 1·4 inches, 2·5 inches, 3·6 inches, 4·7 inches.

2. Measure with your own scale the following lengths :

P _____ Q

R _____ S

3. Draw a straight line AB 4 inches long. Divide an end inch into 8 equal parts. From the scale show how to draw lengths representing 2 miles 3 fur., 3 miles 5 fur., 1 mile 7 fur., the scale being 1 inch to the mile.

4. Draw a straight line AB 5 inches long. Divide an end inch into three equal parts. Show how to read off lengths, from the scale, 4 yds. 1 foot ; 3 yds. 2 feet. (Scale 1 inch to the yd.)

5. If a square mile is represented by a square inch show the space representing 10 acres according to the above scale.

§ 14. Diagonal Scale. A hundredth of an inch is a tenth of a tenth of an inch. If a tenth of an inch is divided into ten equal parts the parts cannot be easily distinguished however sharp your pencil may be. So, to measure lengths to hundreths of an inch we construct a scale called the *diagonal scale* as follows :—

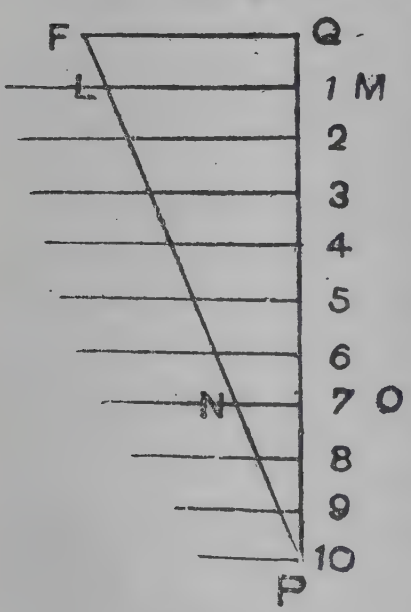

FIG. 11.

Take FQ, a tenth of an inch, draw ten parallels to FQ at equal distances. (The figure given here is a magnified one). From Q draw QP ⊥ to FQ meeting the tenth parallel at P Join PF. Now the △s FQP and LMP are

equiangular and therefore the sides about equal angles are proportional.

$$\therefore \frac{LM}{FQ} = \frac{MP}{QP} = \frac{9 \text{ parts}}{10 \text{ parts}} = \frac{9}{10}$$

$$\therefore LM = \tfrac{9}{10} FQ.$$

But FQ $= \frac{1}{10}$ of an inch.

$\therefore$ LM $= \frac{9}{10} \times \frac{1}{10}$ or $\frac{9}{100}$ of an inch. Similarly the line parallel to LM through 2 $= \frac{8}{100}$ of an inch. Similarly NO $= \frac{3}{100}$ of an inch. Going up the ladder FQ from the foot you will find that the portions of the parallels intercepted between FP and QP increase by a hundredth of an inch.

Take AB an inch long. Divide it into tenths. On AB construct a square ABED. Divide AD also into tenths and through the points of division draw parallels to AB. Join the point E to F the first of the tenth divisions in AB. Through each of the other tenth divisions draw slanting lines (diagonals) parallel to FE. Now the distance between the two marks *pq* will be found to be 7 tenths + 9 hundredths of an inch (= ·79 in.) For the mark *p* is on the line passing through 7 in DE, and is on the line passing through 9 in DA.

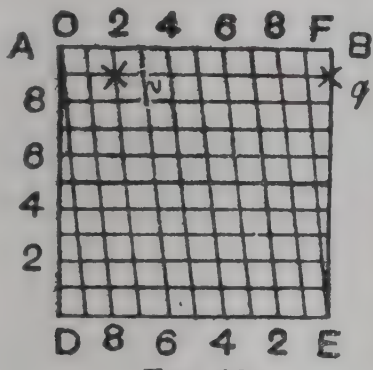

FIG. 12.

$\therefore$ *pq* = 7 tenths + 9 hundredths of an inch.

Exercise II (c).

1. Mark in a similar way on your own figure distances 3 tenths, + 6 hundredths of an inch, ·69 in., ·38 in., ·47 in.

2. Measure the following distances using the diagonal scale.

A_____B

C_____D

3. In constructing your diagonal scale make AD unequal to AB. Examine if that figure will serve the same purpose as well.

4. Draw the accompanying figure for yourself and mark on it the distances 0·8 in., 1·34 in., 2·67 in., 4·93 in., 3·85 in.

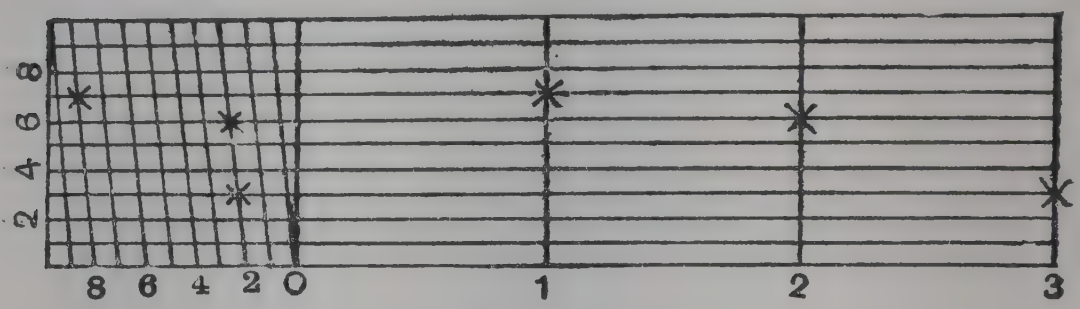

FIG. 13.

5. What is the distance between the two star marks on each of the three horizontal lines of figure 13 ?

6. Convert, by actual measurement, the following metric lengths into inches and decimals of an inch :—2 cm., 3 cm., 4 cm., 5 cm., 6 cm.

Note.—On the same principle a scale can be constructed representing yards, feet and inches. Take one inch to represent a yard. Then a third of an inch will represent a foot, this third if divided into 12 equal parts, by using the principle of diagonal scale, will represent inches. The scale shown below will thus enable us to read any distance in yards, feet and inches if the scale used is one inch to the yard. The distance between the marks in the diagram represents 3 yards 1 ft. 8 inches.

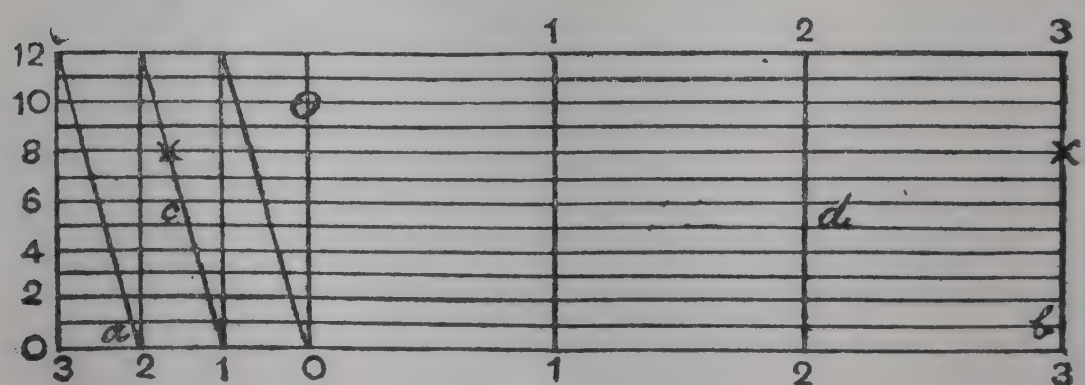

FIG. 14.

7. Mark in a similar manner on the figure distances representing 2 yds. 2 ft. 10 inches and 3 yds. 2 ft., 11 inches.

8. What distances are represented by the lines *ab*, *cd* in the figure ?

§ 15. Representative Fraction. If we draw a plan of a rectangular room 60 ft. by 30 ft. to the scale of 10 ft. to an inch, an actual distance of 10 ft. or 120 inches is represented by 1 inch. In other words, all distances in the plan are $\frac{1}{120}$ of the real distance. The fraction $\frac{1}{120}$ is called the **Representative Fraction** of the plan, usually abbreviated into R. F. The R. F is usually given as a fraction with *unity* as the numerator. For instance, if the scale were 10 inches to the yard

$$\text{R.F.} = \frac{10 \text{ inches}}{1 \text{ yd.}} = \frac{10 \text{ inches}}{36 \text{ inches}} = \frac{10}{36} = \frac{1}{3{\cdot}6}.$$

Exercise II (d).

1. Find the R. F. in each of the following cases :

(1) A plan on a scale of 1 inch to the yard.

(2) ..... 1 inch to the mile.

(3) ..... $\frac{1}{10}$ inch to the furlong.

2. A triangular field ABC is found to have the following measurements :

AB = 60 yds.
BC = 150 yds.
AC = 160 yds.

Make a plan of the field, scale 20 yds. to an inch, and find the length of the perpendicular drawn from A on BC.

3. In a map of a fourth of an inch to the mile what is the R. F., and what length on the map will represent 4000 yds. ?

4. On a French map 1 cm. represents a distance of 1 kilometre. Find the R. F. of the map. What length on the map will represent 2 miles ?

5. Draw a straight line $4\frac{1}{2}$ in. long to represent 5 miles. Construct on this line a plain scale showing miles and furlongs to measure distances up to 5 miles ?

§ 16. Graphical Multiplication and Division.

Ex. 1.—Multiply 5 by 3 graphically.

Let x be the required product

$$5 \times 3 = x \text{ or } 1 \times x. \quad \therefore \quad \frac{5}{1} = \frac{x}{3}.$$

On squared paper take any point O as the origin. Along OX mark off OP and OQ equal to 1 unit and 3 units respectively and along OY mark off OM = 5 units. Through Q draw QN parallel to MP. [Fig. 15 is not drawn to scale.]

Then $\frac{ON}{OQ} = \frac{OM}{OP}$, *i.e.*, $\frac{ON}{3} = \frac{5}{1}$ $\therefore$ ON $= 5 \times 3$.

When measured ON will be found to contain 15 units. Thus graphically you find the product to be 15.

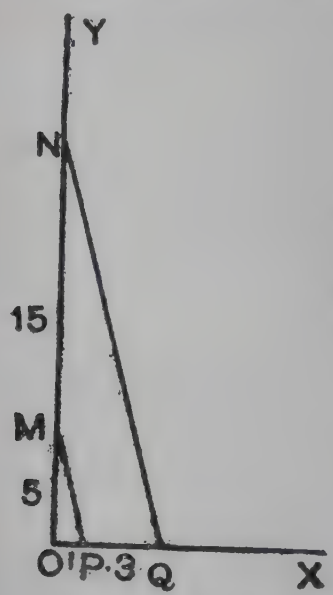

FIG. 15.

Ex. 2. Find graphically $5 \times 3 \times 6$.

We have found ON $= 5 \times 3$.

Now we must represent $5 \times 3 \times 6$.

In the same figure cut off OR equal to 6 units along the x axis. Through R draw RS parallel to PN meeting the y axis at S [The lines are not shown in the figure.[

Then $\frac{OS}{OR} = \frac{ON}{OP}$

i.e., $\frac{OS}{6} = \frac{5 \times 3}{1}$.

$\therefore$ OS $= 5 \times 3 \times 6$

In this way products can be read off along the y axis.

Ex 3.—Find the value of $\frac{81{\cdot}4}{67{\cdot}3}$ graphically.

Now let $\frac{81{\cdot}4}{67\,3} = x$, *i.e.* $\frac{x}{1}$.

Measure along OX and OY, lengths OP and OQ = 67·3 and 81·4 units respectively (take $\frac{1}{10}''$ as the unit). Measure OA = 1 unit along OX. Draw AB parallel to PQ cutting OQ in B.

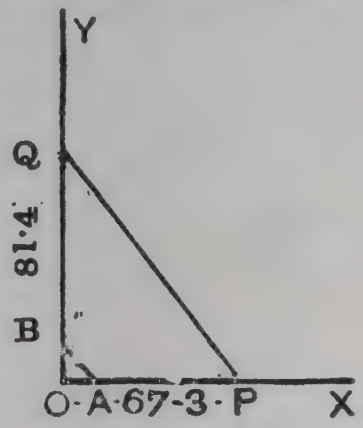

FIG. 16.

Then $\frac{OB}{OA} = \frac{OQ}{OP}$,

$\frac{OB}{1} = \frac{81{\cdot}4}{67{\cdot}3}$.

And OB can be measured.

Ex. 4.—Find graphically the value of $\frac{81{\cdot}4}{67{\cdot}3} \times 5{\cdot}40$.

Let $\frac{81{\cdot}4}{67{\cdot}3} \times 5{\cdot}40 = x$

then $\frac{81{\cdot}4}{67{\cdot}3} = \frac{x}{5{\cdot}40}$.

In Fig. 16 make OA = 5·40 instead of 1 and the result can be easily obtained.

Exercise II (e).

1. Multiply graphically 8 by 16 and 18 by 6.
2. Divide graphically 144 by 9 and 126 by 7.
3. Work out graphically the following as accurately as you can :

$$\frac{37{\cdot}3}{26{\cdot}8}\,;\ \frac{37{\cdot}3}{26{\cdot}8} \times 3{\cdot}92\,;\ \frac{37{\cdot}3}{26{\cdot}8} \times 2{\cdot}68.$$

4. Simplify graphically.

(*a*) ·87 × ·93 × ·36. (*b*) ·63 × ·45 × ·08.

(*c*) (·37 + ·23) ÷ ·14. (*d*) $\frac{{\cdot}64 \times {\cdot}96}{{\cdot}82 \times {\cdot}28}$.

5. Solve graphically the equations ;

(*a*) $\frac{x}{1{\cdot}6} = \frac{7{\cdot}8}{3{\cdot}4}$. (*b*) $\frac{3{\cdot}9}{x} = \frac{4{\cdot}6}{2{\cdot}3}$.

(*c*) $\frac{3{\cdot}86}{2{\cdot}73} = \frac{x}{1{\cdot}25}$. (*d*) $\frac{6{\cdot}39}{8{\cdot}45} = \frac{14}{x}$.

§ **17. Drawing to scale.** To make a copy of a given straight line to a given scale, *i.e.*, to enlarge it or to reduce it in a given ratio.

Let AB be the given straight line. It is required to draw another straight line $= \frac{5}{3}$ times AB, (*i.e*, to enlarge it in the ratio 3 : 5).

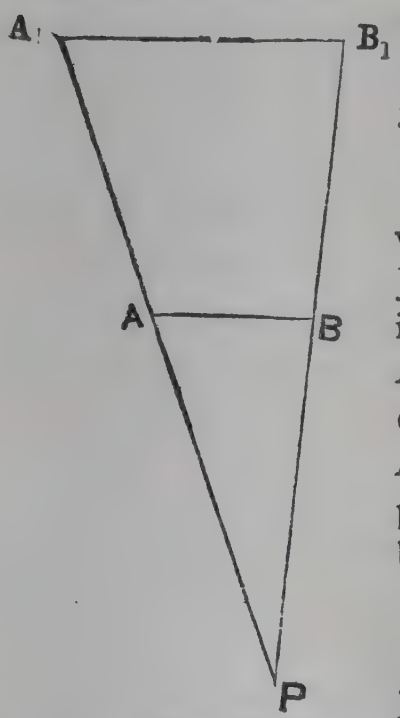

FIG. 17.

Take any point P in any convenient position outside the line. Join PA and PB and divide PA into 3 equal parts and produce it to A_1 so that PA_1 may contain 5 such equal parts. Through A_1 draw A_1B_1 parallel to AB meeting PB produced in B_1. Then A_1B_1 shall be the required line.

$\because$ A_1B_1 is parallel to AB, the $\triangle$s PAB and PA_1B_1 are equiangular and similar.

$$\therefore \quad \frac{A_1B_1}{AB} = \frac{PA_1}{PA} = \frac{5}{3}.$$

$$\therefore \quad A_1B_1 = \tfrac{5}{3}\ AB.$$

The result may also be verified by measurement.

(2) Let AB be a given straight line. It is required to reduce it in the ratio of 4 : 3, *i.e.* to draw another line equal to $\frac{3}{4}$ AB.

Take a point P in any convenient position. Divide PA into 4 equal parts.

Take a point A′ in PA such that PA′ may contain 3 of the equal parts of PA.

Draw A′B′ parallel to AB meeting PB in B′

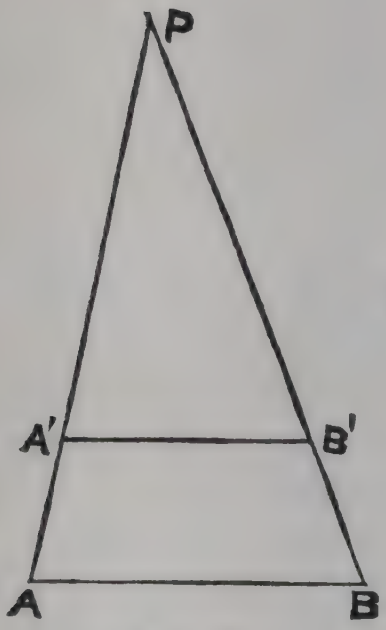

FIG. 18.

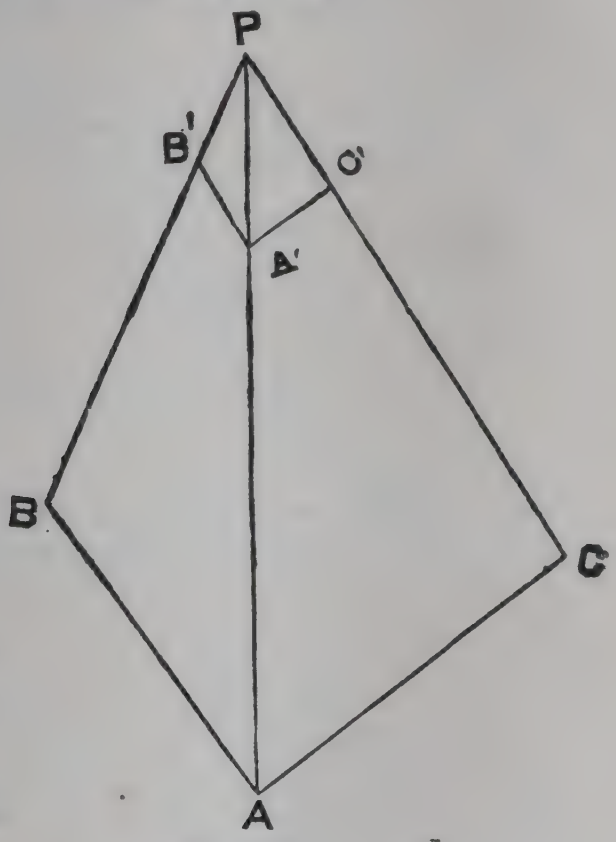

FIG. 19.

Then A′B′ is the required line.

∵ A′B′ is parallel to AB,

∴ △s PA′B′ and PAB are equiangular and therefore similar.

$$\therefore \frac{A'B'}{AB} = \frac{PA'}{PA} = \frac{3}{4}.$$

$$\therefore A'B' = \tfrac{3}{4}\,AB.$$

The result may be verified by measurement.

(3) Make a copy, to a given scale, of two lines making an angle.

Let AB and AC meet at A making the $\angle BAC$. It is required to draw them to a given scale, say $\frac{2}{7}$, cutting at the same angle.

Take a point P in any convenient position. Join PB, PA and PC. Divide PA at A′, so that PA′ may be $\frac{2}{7}$ PA.

Draw A′B′ parallel to AB meeting PB in B′ and A′C′ parallel to AC meeting PC in C′. Then B′A′C′ will be a copy of BAC to the scale $\frac{2}{7}$.

$\because$ A′B′ is parallel to AB,

we have $\frac{A'B'}{AB} = \frac{PA'}{PA} = \frac{2}{7}$.

$\therefore$ $A'B' = \frac{2}{7}$ AB. Similarly $A'C' = \frac{2}{7}$ AC.

Also $\angle B'A'P = \angle BAP$, $\because$ AB and A′B′ are parallel, and $\angle PA'C' = \angle PAC$, $\because$ A′C′ and AC are parallel, $\therefore$ the whole $\angle B'A'C' =$ the whole $\angle BAC$.

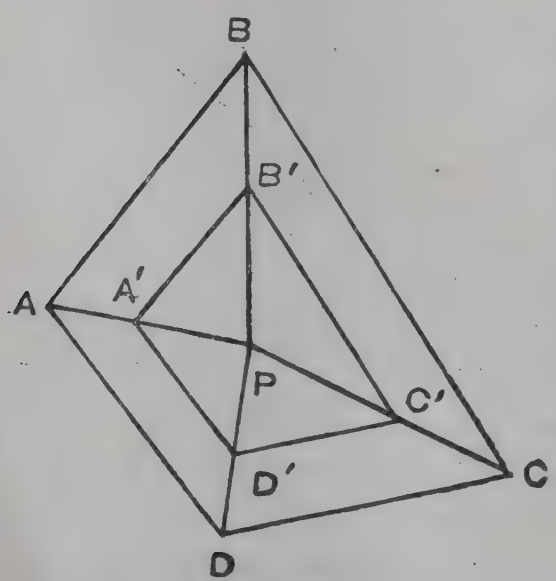

FIG. 20.

The result may be verified by measurement.

(4) To make a copy of a given figure enlarged or reduced in a given ratio.

Let ABCD be a quadrilateral. It is required to draw a quadrilateral similar to ABCD with the sides each $\frac{2}{3}$ of the corresponding side of ABCD.

Take any point P in a convenient position as in Fig. 20 or 21. Join PA, PB, PC, PD.

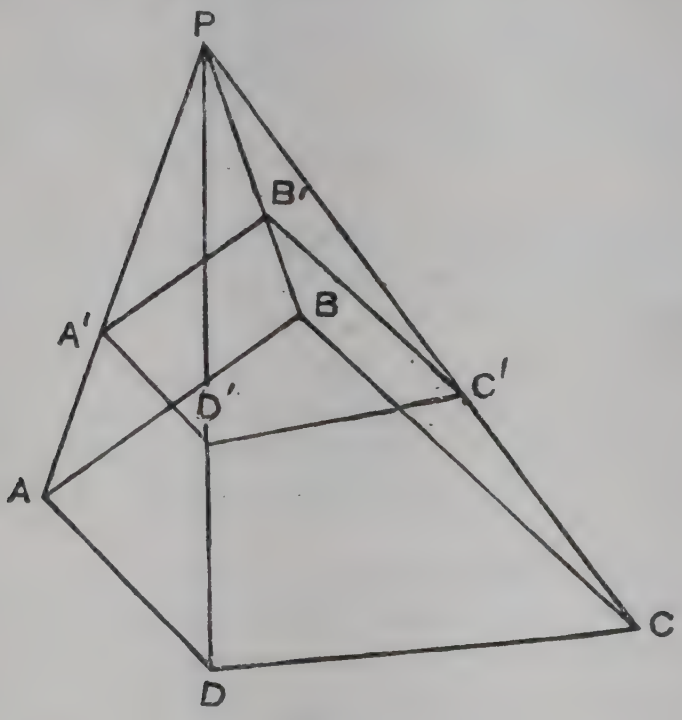

FIG. 21.

Take a point A′, in PA, such that $\frac{PA'}{PA} = \frac{2}{3}$.

Draw A′B′ parallel to AB meeting PB in B′, B′C′ parallel to BC meeting PC in C′, and A′D′ parallel to AD meeting PD in D′ and join C′D′. Then A′B′C′D′ is a copy of ABCD to the scale $\frac{2}{3}$.

The proof is similar to that in Ex. 3 and is left as an exercise to the student.

The result may be verified by measurement.

In the same way figures may be enlarged in a given ratio.

Exercise II (f.)

1. Given a triangle ABC, draw a similar triangle (scale $\frac{1}{3}$) by taking the point P inside the triangle. Measure the lines in your drawing and verify.

2. Given a triangle ABC, draw a similar triangle (scale $\frac{1}{2}$) by taking the point P outside the triangle. Verify by measurement.

3. Draw a rectangle ABCD 4 inches by 3 inches. Make a copy of this reduced to $\frac{1}{6}$. Measure your copy and verify your work. If your copy be named A'B'C'D', draw the diagonals AC and A'C', and show that A'C' is $\frac{1}{6}$ AC.

4. Draw an irregular quadrilateral. Make a copy of it to the scale $\frac{1}{5}$.

5. Draw to scale an end wall of a house with gable roof, being given that the width of the wall = 20 ft., total height to ridge = $26\frac{2}{3}$ ft., height to eaves = $16\frac{2}{3}$ ft., scale, $\frac{1}{240}$.

6. A field in the shape of a right-angled triangle, the two sides measuring 350 ft. and 120 ft., is divided for building into 7 plots, by lines drawn parallel to the shorter side, and at equal intervals. Draw a plan to scale $\frac{1}{840}$. Measure and find the average length of each plot.

7. A field is in shape a quadrilateral ABCD. AB = 1·5 metres, AD = 1·2 metres, $\angle BAD = 90^0$, $\angle ABC = 120^0$, $\angle CDA = 90^0$. Find by measurement the remaining sides of the quadrilateral, scale $\frac{1}{500}$.

8. The sides of a triangular field measure 144 yds., 84 yds. and 96 yds. respectively. Make a plan of this on a scale 1 to 2016.

9. Copy the adjoining figure to scale $\frac{2}{5}$.

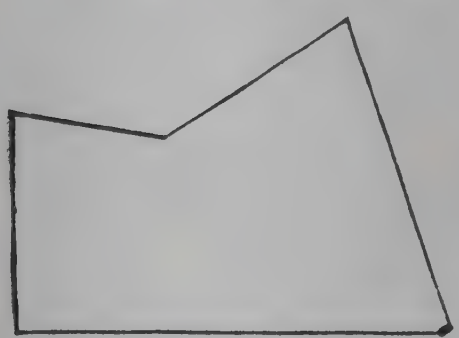

FIG. 22.

10. Enlarge the same figure in the ratio 2 : 7.

11. A rectangular plot of land is purchased, 80 ft. by 24 ft. It is proposed to construct on it a building of the following description. A verandah 6 ft. wide in front running along the entire width of the plot, 2 rooms adjoining the verandah with a passage between, the width of the rooms being 10 ft., two compartments each being 24 ft. by 20 ft.; the compartments are to be separated by two rooms and a passage running between them

exactly equal in size to the front two rooms and the front passage. and opposite to them in position. Draw a plan, scale 1 to 240.

12. An ⊏⊐-shaped enclosure is intended for a cattle-shed. The outer edges are 660 ft. and 198 ft. respectively and the inner edges are 594 ft. and 154 ft. respectively. Draw a plan to scale 1 to 2772. Find the area in acres.

13. The R. F. of a map is $\frac{1}{14400}$. If the R. F. of a new drawing of the same map is to be $\frac{1}{26800}$, to what size is it reduced?

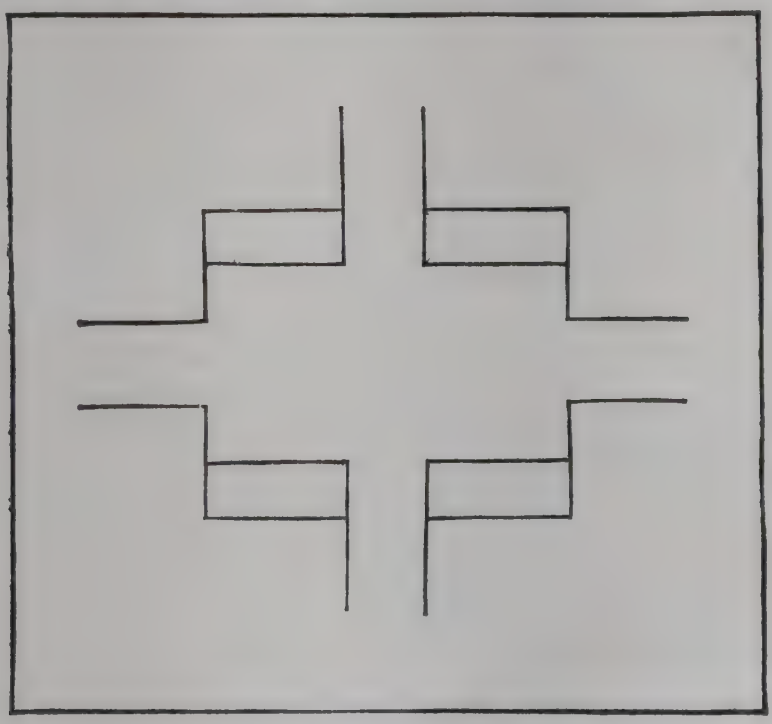

FIG. 23.

14. The above figure is the plan of a garden and of a bungalow in it, scale 1 in 1440. What is the area of the hall in the bungalow and what fraction is it of that of the garden?

Exercise II (g)

1. There are two set squares (each 60°, 30°). Their least sides are 2·3″ and 4·7″. Find, by drawing, the length of the hypotenuse of the one, and then by calculation that of the other.

2. Two equilateral $\triangle$s are cut out of the same zinc plate, their sides are in the ratio of 14·35 : 7·79. If the weight of the former is 3 lbs., what is the weight of the latter ?

3. Construct an angle of 15° with the help of your protractor. Find (1) the sine of the angle, (2) the cosine of the angle, (3) the tangent of the angle.

4. There is a set square (15°—75°). The hypotenuse measures 5·75 inches. Find the shortest side of the set square using the sine of the angle obtained in question 3 and find also by measurement.

5. Find graphically, a fourth proportional to 1000, 1050, 1750, scale 500 to one inch.

6. A room 20 ft. by 16 ft. with a gable roof, has a verandah outside it, running parallel to the length of the room. [The roof over the verandah is in continuation of the roof over the room.] Total height to ridge is $15\frac{1}{3}$ ft. and the greatest and least heights of the verandah are 10 ft. and 6 ft. Find by calculation the width of the verandah and check your result by measurement.

7. A statue stands on a pedestal which is 4·8 feet high. The shadow of the statue thrown by the sun is $18\cdot\dot{6}$ ft. long and of the pedestal 6·6 ft. What is the total height of the statue and pedestal ?

8. A man 5 ft. 6 in. high is $13\frac{1}{2}$ ft. away from the point directly under a lamp. His shadow thrown by the lamp is $4\frac{1}{2}$ ft. long. Find the height of the lamp from the ground. The man moves away to a point distant 20 ft. 3 inches from directly under the lamp. What is the length of his shadow in this new position ?

9. The angle of elevation of the top of a tower when observed from A is equal to the angle of elevation of a cliff from B. Show that the heights of the tower and the cliff are proportional to their distances from A and B respectively.

10. The angle of elevation of the top of a tower of height 150 ft. is 60° as seen from a place A. The observer so moves that the angle of elevation remains unchanged in his different positions. What is his track ?

11. A piece of land in the form of a four-sided figure has two of its sides parallel and of lengths 520 yds. and 400 yds., a third side ⊥ to them and 440 yds. long. The fourth side is a foot-path which is to be moved so as to make the field rectangular without change in area. Draw the field using 1 cm. to represent 40 yds. and show how the fourth side must be altered. What is the area of the field in acres?

12. An enlarged photograph ABC of Fig. 24 is to be made, *ac* being represented by a length (AC) 8 cm. long. Copy the figure and then draw the enlargement denoting the points corresponding to *a*, *b*, *c*, etc., by A, B, C, &c.

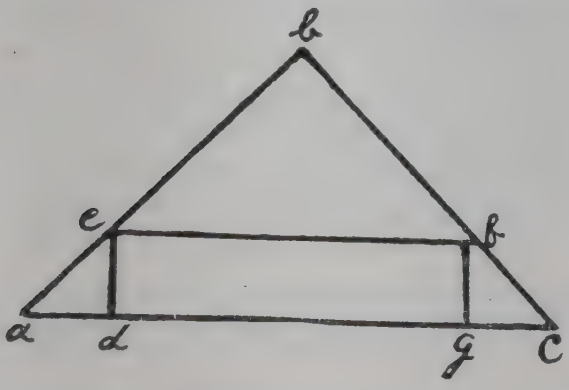

FIG. 24.

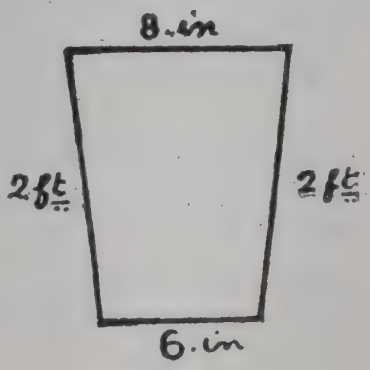

FIG. 25.

13. A semi-circular arch is built of stones cut to the shape shown in Fig. 25. [Fig. 25 is not drawn to scale]. What is the internal diameter of the arch and how many stones are required to complete it?

14. Draw an angle whose sine is ·6 and find the value of its cosine and tangent.

15. A picture is hung in a vertical plane with the top horizontal by a cord 3 ft. long fastened to the two top corners and passing over a nail each part of the string making an ∠ of 60° with the top of the picture. How much will the picture be raised by shortening the cord by 1 ft. and adjusting the picture so that the top is still horizontal? If the picture is to be lowered by 6 inches, what must be the length of the cord?

16. Make an angle whose tangent is ·64. Find the sine and cosine of the angle.

17. To find the breadth of a river with straight banks, two points A and B are taken on one bank, 120 yds. apart. From these a point C on the opposite bank is observed and it is found that $\angle ABC = 40°$ and $\angle BAC = 64°$. Draw a figure on a scale of 1 inch to 40 yds. and find from it the breadth of the river.

18 Draw a plain scale of yards, R.F. $\frac{1}{6\,8\,0}$ to show 100 yards.

19. Make a plain scale of miles and furlongs to show 8 miles for a map with R.F. $\frac{1}{8\,0\,0\,0\,0}$.

20. On a map three points, A, B, C representing three mountain tops lie in order in a straight line, and the heights are stated to be 5300, 4902, 4630 ft. respectively. Given that $AB = 6\frac{1}{2}$ inches and $BC = 1\frac{2}{3}$ inches ; determine whether the top of C can be seen from the top of A.

21. A man walks 300 miles from A to B, turns at right angles to the right and walks 350 miles to C, then turns to the right through an angle of 105° and walks 325 miles to D. Make a plain scale of miles, R,F. $\frac{1}{6\,0\,0\,0}$ and thence draw a plan of the walk. Find the area of ABCD.

22. On a map $54\frac{1}{2}$ miles are represented by 1 foot 2 inches. Find the R.F. and draw a diagonal scale of miles and furlongs showing 30 miles.

23. Draw a diagonal scale of feet and inches showing 30 ft. to suit a plan whose R.F. is $\frac{1}{6\,0}$. Indicate by two small marks on the scale a distance of 14 ft. 8 inches.

24. A person walks 180 yds. westwards, and then 250 yds. in a direction 50° northwards of west. Draw a figure to any convenient scale and determine by measurement how many yards he is to the west and how many to the north of the position from which he started.

CHAPTER III.

IDENTITIES AND FACTORS.

* Pupils should be asked to fold paper or to cut thin cardboard so as to perceive the truth of each of the identities in this chapter.

§ 18. **$p(x+y+z)=px+py+pz$.** We have already learnt in the Chapter on Multiplication (Part I) that $p(x+y+z)=px+py+pz$. We shall in this chapter presently see how such algebraical formulæ can be geometrically interpreted and how such formulæ will be useful in finding factors of expressions in Algebra.

ADHE is a rectangle divided by lines BF and CG each parallel to AE. Let AE, AB, BC and CD contain p, x, y and z units of length respectively.

The rectangle AH equals AD × AE, *i.e.*, contains $p(x+y+z)$ units of area.

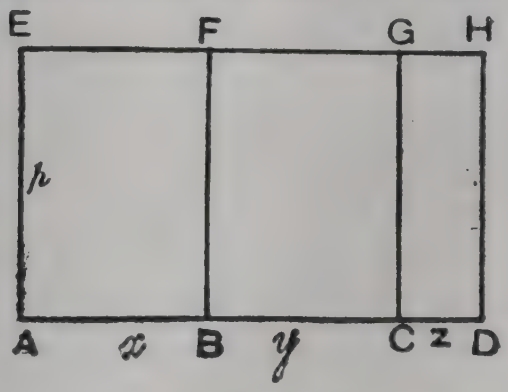

FIG. 26.

Similarly the rectangles EB, FC, GD contain respectively px, py and pz units of area.

And the whole rectangle AH is equal to the sum of the rectangles EB, FC, GD

$\therefore\ p(x+y+z)=px+py+pz$ I

Similarly it may be shown that $p(x+y+z+u+v+.....)$

$=px+py+pz+pu+pv+$ II

I and II may be geometrically interpreted thus :

"If there are two straight lines, one of which is divided and the other undivided, the rectangle contained

by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line."

N.B.—A rectangle is said to be contained by any two of its adjacent sides and the area of a rectangle is equal to the product of the length and breadth of the rectangle.

§ 19. $(x + y)(x + z) = x^2 + xy + xz + yz.$

We know that $p(x+z)=px+pz$.

If $p=x+y$, we have $(x+y)(x+z)=(x+y)x+(x+y)z$

$=x^2+xy+xz+yz=x^2+x(y+z)+yz$ III

which may be geometrically illustrated as in Fig 27.

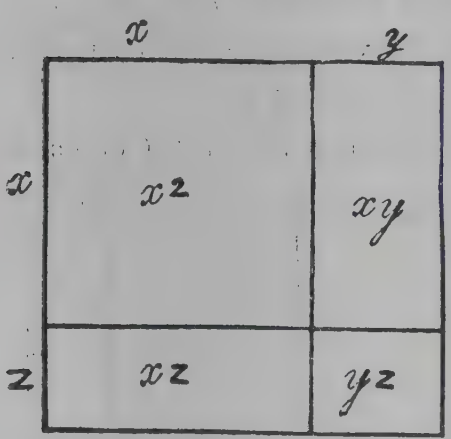

FIG. 27.

To *factorize* an algebraical expression means to express it as the product of factors.

Ex. 1. Factorise the following expressions :—(i) $7x + 7y$; (ii) p^2+pq ; (iii) $a^2+ab+ac$.

(i) $7x+7y$. Here 7 is a factor of each term, hence $7x + 7y = 7(x+y)$.

(ii) $p^2+pq=p.p+p.q=p(p+q)$.

(iii) $a^2+ab+ac=a.a+a.b+a.c=a(a+b+c)$.

Ex. 2. Find the value of $27^2+27\times73$

$27^2+27\times73=27(27+73)=27\times100=2700$.

Exercise III (a).

Factorise :—

1. a^2+ab. 2. b^2+2bc. 3. $p^2+2pq+2pr$.
4. a^2b+a^2c+abc. 5. $a^2bc+ab^2c+abc^2$.
6. Show by a geometrical figure that $a(b-c)=ab-ac$.
7. Show by a geometrical figure that $a(b-c-d)=ab-ac-ad$.
8. Show by a geometrical figure that $a(b+c-d)=ab+ac-ad$.

Factorise :—

9. $a^2-ab-ac$. **10.** $2b^2+2bc-2bd$.

11. a^2b-ab^2-abc. **12.** $x^2yz+xy^2z-xyz^2$.

§ 20. $(x+y)^2=x^2+2xy+y^2$. ... ... IV.

$(x + y)(x + y)$

$= (x + y)x + (x + y)y$

$= x^2 + yx + xy + y^2$,

i.e., $(x + y)^2 = x^2 + 2xy + y^2$

The same can be derived from III by putting $z = y$.

ABDE is a square divided by lines CR, FG paralell to the sides. Also EF = AC and $\therefore$ FA = CB.

Let AC, CB contain x and y units of length respectively. Then AB contains $(x+y)$ units of length.

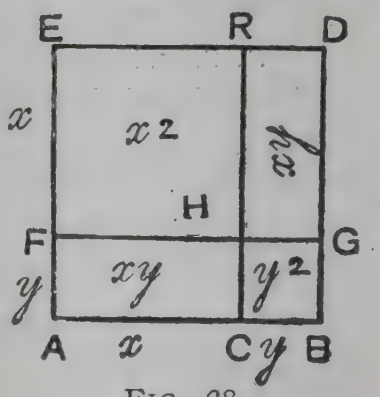

FIG. 28.

Figure ABDE=fig. FR+fig. RG +fig. GC+fig. CF.

But fig. ABDE = AB.AE = $(x+y)(x+y)$ or $(x+y)^2$ units of area.

fig. FR=EF.ER=$x.x$ or x^2 ,, ,,

fig. RG=RD.DG=$y.x$ or xy ,, ,,

fig. CG=BG.BC=$y.y$ or y^2 ,, ,,

and fig. CF=AC.AF=$y.x$ or xy ,, ,,

$\therefore (x+y)^2=x^2+xy+y^2+xy=x^2+2xy+y^2$.

The formula may be geometrically interpreted thus : **"If a straight line is divided into two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts."**

Identity IV gives the expansion of the square of a binomial (an expression containing two terms) and may be stated as follows :

The square of the sum of two terms or numbers is equal to the sum of the squares of the terms together with twice the product of the terms.

Ex. 1. Find the square of $2a + 3b$.

$(2a + 3b)^2 = (2a)^2 + (3b)^2 + 2.2a.3b = 4a^2 + 9b^2 + 12ab$.

Ex. 2. Examine if $a^2 + 10a + 25$ is a perfect square and express it as a square.

$a^2 + 10a + 25 = (a)^2 + (5)^2 + 2.a.5 = (a + 5)^2$.

Ex. 3. What term must be added to $81a^2 + 64b^2$ to make the whole a perfect square?

$81a^2 + 64b^2 = (9a)^2 + (4b)^2$.

$\therefore$ $2.9a.4b$ should be added, *i.e.*, $72ab$.

Then the whole becomes $(9a + 4b)^2$.

Ex. 4. What term must be added to $81a^2 + 72ab$ to make the whole a perfect square.

$81a^2 + 72ab = (9a)^2 + 2.9a.4b$.

$\therefore$ $(4b)^2$ must be added.

Exercise III (b).

Write the squares of—

1. $(x + 3)$. **2.** $(y + 6z)$. **3.** $(1 + 3a)$.

4. $(ab + ac)$. **5.** $(a^2 + ab)$. **6.** $(10p + 3r)$.

7. Find what terms must be added to each of the following so as to make the whole a perfect square :—

(*a*) $9x^2 + 16y^2$. (*b*) $49a^2 + 225b^2$.

(*c*) $a^2b^2 + b^2c^2$. (*d*) $9a^2 + 48ab$.

(*e*) $a^2 + 2500$. (*f*) $16y^2 + 64xy$.

(*g*) $a^2 + ab$. (*h*) $b^4 + 16$.

8. Find, by using the formula IV, the value of—

(*a*) 13^2 or $(10 + 3)(10 + 3)$. (*b*) 12^2 or $(8 + 4)(8 + 4)$.

(*c*) 15^2 or $(7 + 8)(7 + 8)$. (*d*) $(x + 8)(x + 8)$.

9. If $x^2 + 2qx + 36$ is a perfect square, what is the value of q?

10. Break up the following square numbers into parts as suggested by the formula. $(a + b)^2 = a^2 + 2ab + b^2$:—64, 169, 225, 625.

11. Simplify with as little actual multiplication as possible $(1001)^2$, $(1005)^2$, $(1003)^2$, $(612)^2$.

§ 21. $\mathbf{(x-y)^2=x^2-2xy+y^2}$ V

$(x-y)(x-y)$

$= (x-y)x-(x-y)y$

$= (x^2-yx)-(xy-y^2)$

$= x^2-xy-xy+y^2,$

i.e., $(x-y)^2 = x^2-2xy+y^2.$

ACDE is a square divided by lines BK, FH parallel to the sides. Also AF = AB, and ∴ FE = BC. Let AC, BC contain x and y units of length respectively.

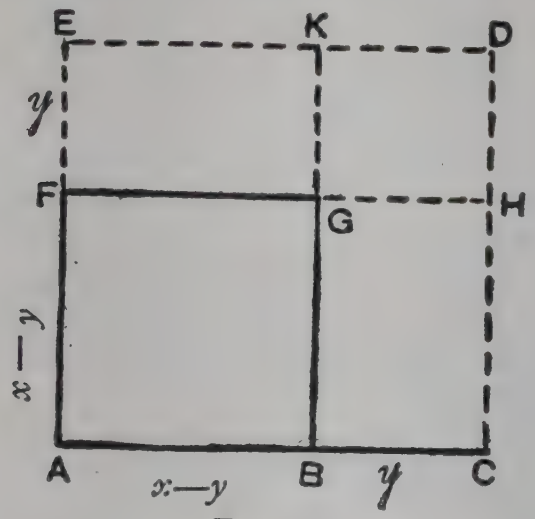

FIG. 29.

Fig. AG = fig. AD — fig. EH — fig. GC.

= fig. AD — fig. EH — fig. GC — fig. KH + fig. KH.

= fig. AD — fig. EH — fig. KC + fig KH.

But fig. AG = AB.AF = $(x-y)(x-y)$ or $(x-y)^2$ units of area.

fig. AD = AC. AE = $x.x$ or x^2 " "

fig. EH = FE. FH = $x.y$ or xy " "

fig. KC = BC. CD = $y.x$ or xy " "

fig. KH = KG. GH = $y.y$ or y^2 " "

$$\therefore\ (x-y)^2 = x^2 - xy - xy + y^2$$
$$= x^2 - 2xy + y^2.$$

The formula may be geometrically interpreted thus :

"If a straight line be produced to a point then the square on the line is equal to the sum of the squares on the whole line thus produced and the part produced diminished by twice the rectangle contained by the whole line produced and the part produced."

Identity V gives the expansion of the square of a binomial being the difference of two terms and may be stated as follows :

"The square of the difference of two terms or numbers is equal to the sum of the squares of the terms minus twice the product of the terms."

Ex. 1. Find the square of $a - 3b$.

$(a-3b)^2 = (a)^2 + (3b)^2 - 2a.3b = a^2 + 9b^2 - 6ab.$

Ex. 2. Examine if $a^2 - 10a + 25$ is a perfect square and express it as a square.

$a^2 - 10a + 25 = (a) + (5)^2 - 2.a.5 = (a-5)^2.$

Ex. 3. What term must be subtracted from $81a^2 + 64b^2$ to make the resulting expression a perfect square ?

$81a^2 + 64b^2 = (9a)^2 + (4b)^2.$

$\therefore$ $2 \cdot 9a.4b$, *i.e.*, $72\,ab$.

$\therefore$ then the resulting expression is equal to $(9a - 4b)^2$.

Ex. 4. What term must be added to $25a^2 - 40ab$ to make the whole a perfect square ?

$25a^2 - 40ab = (5a)^2 - 2.5a.4b.$

$(4b)^2$ must be added.

The general principles involved in Ex. 3 and 4 are exactly the same as those mentioned in the corresponding exercises worked out under the formula $(a+b)^2 = a^2 + 2ab + b^2$.

Exercise III (c).

Write down the squares of—

1. $(x-3)$.	**2.** $(y-8)$.	**3.** $(3a-4b)$.
4. $(12ab-1)$.	**5.** $(9a-7)$.	**6.** $(xy-xz)$.

Find what terms must be subtracted from each of the following to make the resulting expression a perfect square :—

7. $100a^2 + 36b^2$. **8.** $49x^2y^2 + 64x^2z^2$.

9. $x^4 + 4y^4$. **10.** $b^2c^2 + 4a^4$.

11. $25 + 144a^2$.

Find what terms must be added to each of the following to make the resulting expression a perfect square :—

12. $x^2 - 4xy$. **13.** $16a^2 - 40abc$.

14. $9x^2y^2 - 54xyz$. **15.** $b^2c^2 - abc$.

16. $25a^2b^2c - 15abc$. **17.** $x^2 - \frac{bx}{a}$.

18. Find, by using the formula V, the value of

(*a*) 8^2 or $(10 - 2)\ (10 - 2)$.

(*b*) 19^2 or $(20 - 1)\ (20 - 1)$.

(*c*) $(p - q)\ (p - q)$.

19. If $x^2 - 2px + 100$ is a perfect square, what is the value of p?

20. Break up the following square numbers into parts as suggested by the formula $(a - b)^2 = a^2 - 2ab + b^2$: 81, 361, 324, 8100.

21. Find the value, with as little actual multiplication as possible, of 998^2, 97^2, 697^2, 9999^2.

§ 22. $x^2 - y^2 = (x+y)(x-y)$ .... VI.

$(x + y)(x-y)$
$= (x + y)x$
$-(x + y)y$
$= x^2 + xy - xy$
$- y^2 = x^2 - y^2$.

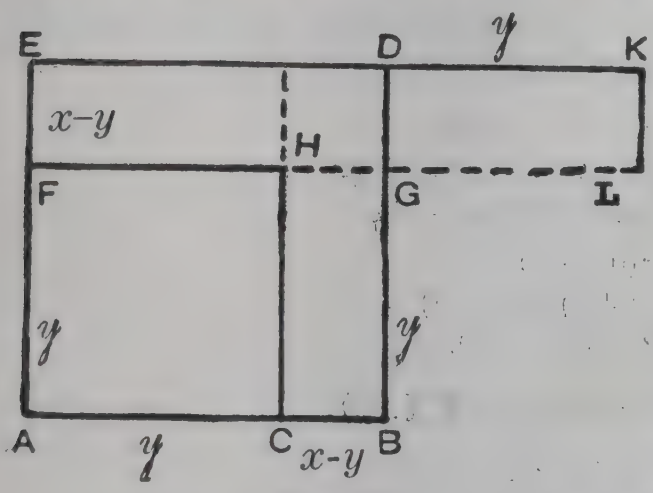

FIG. 30.

ABDE is a square divided internally and externally by lines CH, FL and KL which are parallel to the sides of the square.

Here AC, AF and DK are taken equal and it can be shown that KL = DG = FE = CB.

Let AB, AC contain x and y units of length respectively.
∴ CB contains $x - y$ units of length.

Now fig. AD — fig. AH
= fig. EG + fig. HB
= fig. EG + fig. DL
= fig. EL.

But fig. AD = AB. AE = x. x or x^2 units of area
fig. AH = AC. AF = y. y or y^2 „
and fig. EL = EK. EF = $(x + y)$ $(x - y)$.

∴ $x^2 - y^2 = (x + y)(x - y)$.

The formula may be geometrically interpreted thus:

"The difference between the squares on two given lines is equal to the rectangle contained by the sum and the difference of the two given lines."

Identity VI enables us to find the factors of an expression in the form of the difference of two squares and may be stated as follows:

"The difference of the squares of two terms or numbers is equal to the product of their sum and difference.

Ex. 1. Factorise:— $4a^2 - 9b^2$.

$4a^2 - 9b^2 = (2a)^2 - (3b)^2$
$= (2a + 3b)(2a - 3b)$.

Ex. 2. Multiply 101 by 99

$101 \times 99 = (100 + 1)(100 - 1)$
$= 100^2 - 1^2 = 9999$.

Exercise III (d).

1. Find by formula the value of—

(*a*) 8×12 or $(10 - 2)(10 + 2)$. (*b*) 11×5 or $(8 + 3)(8 - 3)$.
(*c*) 22×18 or $(20 + 2)(20 - 2)$. (*d*) $(x + 1)(x - 1)$.

2. Find the value of :—

(*a*) $(x + a)(x - a)$. (*b*) $(p + q)(p - q)$.

(*c*) $(4x + 3y)(4x - 3y)$. (*d*) $(12x + 5y)(12x - 5y)$.

3. Factorise :—

(*a*) $64 - x^2$. (*b*) $x^2 - 4$.

(*c*) $x^2 - 100$. (*d*) $x^2 - 2500$.

(*e*) $(a + b)^2 - c^2$. (*f*) $49x^2 - 81y^2$.

(*g*) $(x + 1)^2 - 3^2$. (*h*) $(y + 2)^2 - (2y + 1)^2$.

4. Simplify with as little actual multiplication as possible :—

(*a*) $36^2 - 24^2$. (*b*) $100^2 - 98^2$.

(*c*) $(3001)^2 - (2999)^2$. (*d*) $(999{\cdot}5)^2 - (998{\cdot}7)^2$.

(*e*) 998×1002. (*f*) 2001×1999.

(*g*) $1499 \times 501 + 899 \times 99$. (*h*) $699 \times 1701 - 499 \times 1501$.

5. If $x^2 - 100 = (x + p)(x - p)$, what is p ?

§ 23. $(a + b)^2 - (a - b)^2 = 4ab$ ... vii

$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$ ... viii

$(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$.

$\therefore (a + b)^2 - (a - b)^2 = 4ab$, and $(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$.

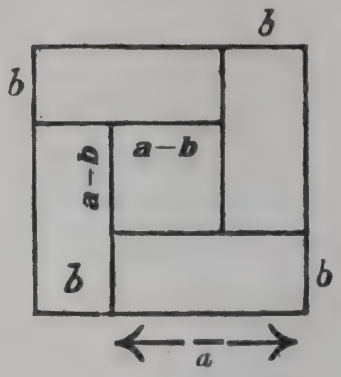

FIG. 31.

These may be geometrically illustrated as follows : —

Fig. 29 is a square which is made up of 4 rectangles and a central square. Each of the rectangles is ab, the central square is $(a - b)^2$ and the whole square is $(a + b)^2$.

$\therefore$ we get

$$(a + b)^2 = (a - b)^2 + 4ab$$

or $$(a + b)^2 - (a - b)^2 = 4ab$$

In Fig. 30, ABCD represents $(a + b)^2$ and EFGH represents $(a - b)^2$.

$$\therefore (a + b)^2 + (a - b)^2 = \text{fig. AC} + \text{fig. FH}$$
$$= \text{fig. HB} + \text{fig. DF} + \text{fig. AE} + \text{fig. GC}$$
$$= a^2 + a^2 + b^2 + b^2$$
$$= 2a^2 + 2b^2.$$

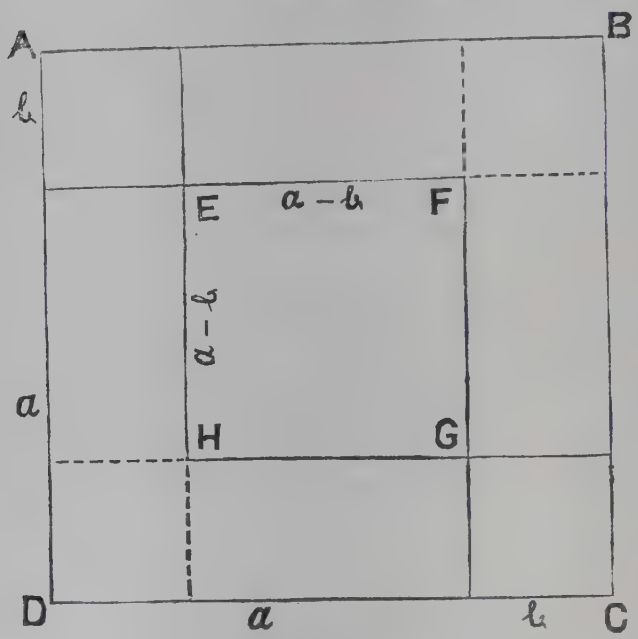

FIG. 32.

$$\therefore (a+b)^2+(a-b)^2 = 2a^2+2b^2.$$

§ **24.** We have seen that $x(p+q) = xp + xq$.

If $x = r + s$,

then $(r + s)(p + q)$
$$= (r+s)\,p + (r+s)q$$
$$= rp+sp+rq+sq.$$

The geometrical illustration of this formula is left as an exercise to the student.

Ex. 1. Factorise $ad+bc+ac+bd$. The expression may be re-arranged thus :— $ac+ad+bc+bd$, so that the first two terms may have a common factor and the next two also.

$\therefore$ the expression $= a\,(c+d) + b\,(c+d)$
$= (c+d)\,(a+b)$.

Ex. 2. Factorise $ac - bc - ad + bd$

$$ac - bc - ad + bd = c(a - b) - d(a - b)$$
$$= (a - b)\,(c - d).$$

Exercise III (e).

Factorise :—

1. $xc + yd + xd + yc$.
2. $xc + yd - xd - yc$.
3. $xc - yd + xd - yc$.
4. $xc - yd - xd + yc$.
5. $8ac - 9bd + 12bc - 6ad$.
6. $xa + x + a + 1$.
7. $xa - x - a + 1$.
8. $a^2 + bc + ca + ab$.

§ **25.** Now $(x+a)(x+b)$

$$= x(x+b) + a(x+b)$$
$$= x^2 + ab + ax + ab$$
$$= x^2 + x(a+b) + ab$$

as is evident from a figure as in Art. 19.

Rule :—When two binomials whose first terms are the same are multiplied, the product consists of three terms, the first term is the square of the first term in each binomial, the second is the product of the first term in the binomial and the sum of the second terms of the binomials and the third term is the product of the last terms of the binomials.

Ex. 1. Expand $(x+3)(x+8)$.

There must be 3 terms in the product.

1st term = square of x, *i.e.* x^2.

2nd term = product of x and the sum of 8 and $3 = 11x$.

3rd term = product of 3 and $8 = 24$,

i.e., $(x+3)(x+8) = x^2 + 11x + 24$.

N.B.—In actual working these operations may be done mentally and the expansion at once written down.

Ex. 2. Expand $(x-3)(x+7)$

$$(x-3)(x+7) = (x-3)x + (x-3)7$$
$$= x^2 - 3x + 7x - 7 \cdot 3$$
$$= x^2 + x(7-3) - 7 \cdot 3.$$

Thus here also the 1st term $= x^2$.

2nd term $= x \times$ (sum of the second terms, *viz.* -3 and $+7$)

3rd term = product of the second terms,
viz. -3 and $+7$.

Thus the above rule holds in this case also.

Ex. 3. Expand $(x-8)(x-7)$.

$$\begin{aligned}(x-8)(x-7) &= (x-8)x - 7(x-8)\\ &= x^2 - 8x - 7x + 7 \cdot 8\\ &= x^2 - x(8+7) + 7 \cdot 8.\end{aligned}$$

Thus here also 1st term $= x^2$.

2nd term $= x \times$ (sum of the second terms,
viz. -8 and -7).

3rd term = product of the second terms,
viz. -8 and -7.

It will thus be seen that the formula $(x+a)(x+b) = x^2 + x(a+b) + ab$ and the corresponding rule holds whether a and b are positive or negative.

Exercise III (f).

Expand by formula :—

1. $(x+1)(x+7)$.	**2.** $(x+8)(x+2)$.
3. $(3x+5)(3x+3)$.	**4.** $(x-8)(x+4)$.
5. $(2x-1)(2x+7)$.	**6.** $(3x+7)(3x-5)$.
7. $(7a+b)(7a+3b)$.	**8.** $(3a+2)(3a-5)$.

Find, by formula, the value of :—

9. 21×27.	**10.** 63×56.
11. 99×103.	**12.** 88×94.

§ **26.** The opposite process of converting an expression containing three terms as stated in the rule into the product of two binomials is equally interesting and probably more important.

Ex. 1. Factorise $x^2 + 5x + 6$ and find for what values of x the expression becomes zero.

Here there are three terms, the first is the square of x, the second is the product of the x and 5. ∴ this 5 should be the sum of the two last terms of the required binomial factors, and the third term 6 should be the product of the two last terms. We can from our knowledge of numbers at once find two numbers whose sum is 5 and whose product is 6. By trial they are found to be 3 and 2. ∴ one binomial is $x + 3$ and the other binomial is $x + 2$.

i.e., the factors of $x^2 + 5x + 6$ are $(x + 3)$ and $(x + 2)$.

In actual practice, all this is mentally done and the following working alone is shown :—

$$x^2 + 5x + 6 = x^2 + 3x + 2x + 6$$
$$= x(x + 3) + 2(x + 3)$$
$$= (x + 3)(x + 2).$$

Now $(x + 2)(x + 3)$ will be zero if either $x + 2$ is zero or $x + 3$ is zero, *i.e.*, if $x = -2$ or $= -3$.

Verify by substituting these values in the expression.

Ex. 2. Solve $x^2 - 3x - 10 = 0$.

First find the factors of $x^2 - 3x - 10$. For this we have to find two numbers whose sum is -3 and whose product is -10.

By trial the numbers are -5 and $+2$,

for $-5 + 2 = -3$ and $-5 \times +2 = -10$.

$$\therefore \quad x^2 - 3x - 10 = x^2 - 5x + 2x - 10$$
$$= x(x - 5) + 2(x + 5)$$
$$= (x - 5)(x + 2).$$

Now the expression $x^2 - 3x - 10 = 0$ if either of the factors $x - 5$ or $x + 2 = 0$, *i.e.*, if $x = 5$ or $x = -2$.

Thus 5 and -2 are the roots of the equation $x^2 - 3x + 10 = 0$.

Verify the roots by substituting these values in the equation.

Exercise III (g).

Factorise :—

1. $x^2-7x+10$.
2. $x^2-11x+10$.
3. $x^2-9x+20$.
4. x^2-x-90.
5. $x^2-2x-15$.
6. $9x^2+24x+15$.

Solve the equations :—

7. $4x^2-8x+3=0$.
8. $x^2+\frac{3}{4}x+\frac{1}{8}=0$.
9. $1+5x+6x^2=0$.
10. $4+2y-6y^2=0$.

§ **27.** Let us find the product of $(3x+4)$ and $(2x+5)$. The product

$$=3x(2x+5)+4(2x+5)$$
$$=6x^2+15x+8x+20$$
$$=6x^2+23x+20.$$

Observe that 23, the coefficient of x, is obtained by adding 15 and 8 and $15 \times 8 = 120$, the same as 6×20.

Example. Expand $(2x-3)(3x-5)$.

$$(2x-3)(3x-5)$$
$$=(2x-3)3x-(2x-3)5$$
$$=6x^2-9x-10x+15$$
$$=6x^2-19x+15.$$

Observe that -19, the co-efficient of x, is got by adding -9 and -10 and $-9 \times -10 = 90$ and 15×6 is also 90.

Generally $(ax+b)(cx+d) = acx^2+adx+bcx+bd$ $= acx^2+x(ad+bc)+bd$.

Observe that the co-efficient of x is the sum of ad and bc, whose product is $ad \times bc$, which is the same as $ac \times bd$, *i.e.*, the same as the product of the co-efficient of x^2 and the last term.

Exercise III (h).

Expand :—

1. $(2x+3)(3x+4)$.
2. $(3x+6)(6x-7)$.
3. $(3x-8)(9x-7)$.
4. $(10x-9)(9x-3)$.
5. $(8x-3)(7x-6)$.
6. $(8x-9y)(4x-5y)$.

Find, by formula, the following products :—

7. 73 × 46.
8. 86 × 67.
9. 105 × 93.
10. 127 × 134.

Factorise :— $10x^2+3x-18$.

If $10x^2 + 3x - 18$ is to be resolved into two factors of the form $(ax + b)$ and $(cx + d)$, evidently we must try to put it into the form $acx^2 + x(ad + bc) + bd$ in which we have observed that the co-efficient of x is the sum of two quantities whose product is the same as the product of the co-efficient of x^2 and the last term, *i.e.*, we must try to express $+ 3$ as the sum of two quantities whose product is equal to 10×-18 or -180. Evidently the quantities are $+15$ and -12 $\because$ their sum is $+3$ and their product is -180 and the expression $10x^2 + 3x - 18$ may be written

$$\begin{aligned} & 10x^2 + x(15 - 12) - 18 \\ = \; & 10x^2 + 15x - 12x - 18, \\ = \; & 5x(2x+3) - 6(2x+3) \\ = \; & (2x+3)(5x-6). \end{aligned}$$

Exercise III (i).

Factorise :—

1. $12x^2+5x-2$.
2. $8x^2+30x+27$.
3. $45x^2+106x+45$.
4. $10a^2+7a-12b^2$.

Solve the equation :—

5. $21x^2 - 17x+2=0$.
6. $36x^2-29x-20=0$.

§ **28.** The following are very important and deserve to be noticed both in multiplication and factorisation:—

I. Multiply a^2+ab+b^2 by a^2-ab+b^2.

$$\begin{aligned}(a^2+ab+b^2)(a^2-ab+b^2) &= (a^2+b^2+ab)(a^2+b^2-ab)\\ &= (a^2+b^2)^2-(ab)^2\\ &= a^4+2a^2b^2+b^4-a^2b^2\\ &= a^4+a^2b^2+b^4.\end{aligned}$$

Hence $\mathbf{(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4.}$

II. Factorise :— m^2+4

$$m^4+4=(m^2)^2+(2)^2.$$

Now $2.m^2.2$ should be added to $(m^2)^2+(2)^2$ to make it a perfect square and the same should be subtracted in order to have the value of the expression unaltered,

i.e., we write $\begin{aligned}m^4+4 &= (m^2)^2+(2)^2+2\ m^2.2-4m^2\\ &= (m^2+2)^2-(2m)^2\\ &= (m^2+2+2m)(m^2+2-2m).\\ &= (m^2+2m+2)(m^2-2m+2)\end{aligned}$

III. Factorise :— $a^4-23a^2b^2+b^4$

$$\begin{aligned}a^4-23a^2b^2+b^4 &= a^4+b^4+2a^2b^2-2a^2b^2-23a^2b^2\\ &= (a^2+b^2)^2-25a^2b^2\\ &= (a^2+b^2+5ab)(a^2+b^2-5ab).\\ &\quad (a^2+5ab+b^2)(a^2-5ab+b^2)\end{aligned}$$

IV. Multiply a^2-ab+b^2 by $a+b$.

$$\begin{aligned}&(a^2-ab+b^2)(a+b)\\ &= a(a^2-ab+b^2)+b(a^2-ab+b^2)\\ &= a^3-a^2b+ab^2+a^2b-ab^2+b^3\\ &= a^3+b^3.\end{aligned}$$

Hence $\mathbf{a^3+b^3=(a+b)(a^2-ab+b^2).}$

V. Resolve :— $8x^3+27y^3$

$$\begin{aligned}8x^3+27y^3 &= (2x)^3+(3y)^3\\ &= (2x+3y)\{(2x)^2-2x.3y+(3y)^2\} \text{ by IV}\\ &= (2x+3y)(4x^2-6xy+9y^2)\end{aligned}$$

VI. Multiply a^2+ab+b^2 by $a-b$.

$$(a^2+ab+b^2)(a-b)=a(a^2+ab+b^2)-b(a^2+ab+b^2)$$
$$=a^3+a^2b+ab^2-a^2b-ab^2-b^3$$
$$=a^3-b^3.$$

Hence $a^3-b^3=(a-b)(a^2+ab+b^2)$.

VII. Resolve x^3-125.

The expression $=(x)^3-(5)^3$
$$=(x-5)(x^2+x.\,5+5^2)$$
$$=(x-5)(x^2+5x+25).$$

Exercise III (j).

Perform the following multiplications with the aid of formulæ :—

1. x^2+x+1 by x^2-x+1.
2. a^2+2a+1 by a^2-2a+1.
3. $a+b+c$ by $a+b-c$.
4. $a-b+c$ by $a-b-c$.
5. $x+1$ by x^2-x+1.
6. $x-1$ by x^2+x+1.
7. $x-2$ by x^2+2x+4.
8. $x+2$ by x^2-2x+4.
9. a^2+2a+2 by a^2-2a+2.
10. a^2+2a+4 by $a-2$.

Exercise III (k).

1. Draw a diagram to illustrate :—
   (*a*) $(x+2)(x+3)=x^2+5x+6$.
   (*b*) $(2x+3)^2=4x^2+12x+9$.
   (*c*) $(3x-5)^2=9x^2-30x+25$.
   (*d*) $4x^2-9=(2x+3)(2x-3)$.

2. Resolve $x^2+10x+24$ into factors ; express this as the difference of two squares and then apply the formula $a^2-b^2=(a+b)(a-b)$ and verify your result by actual multiplication of the factors.

3. Simplify :—

   (*a*) $\dfrac{999\times999-768\times768}{999+768}$.

   (*b*) $\dfrac{\cdot01\times\cdot01-\cdot001\times\cdot001}{\cdot01-\cdot001}$.

   (*c*) $\dfrac{(35)^3-(27)^3}{8}$.

   (*d*) $\dfrac{(10\cdot5)^3+(9\cdot5)^3}{20}$

4. The area of a rectangle is $3a^2-2ab-b^2$ sq. ft. Find out the dimensions of the rectangle in feet.

5. Evaluate without multiplication $1589 \times 91 + 1589 \times 109$.

6. The area of a circle is πr^2 where r is the radius of a circle and $\pi = 3{\cdot}1416$. Find the area of the space between the circumferences of two concentric circles whose radii are 1·785 ft. and 1·779 ft. with as little multiplication as possible.

7. A square tank a side of which is $\frac{1}{4}$ mile long is surrounded by streets on all sides each being 11 ft. wide. Find the area of the streets in two ways by expressing it (1) as the difference of two squares ; (2) as the sum of four rectangles.

8. Simplify with as little multiplication as possible :—

(*a*) $\dfrac{(\frac{3}{4})^2 + \frac{3}{4}.\frac{13}{16} - \frac{7}{128}}{(\frac{3}{4})^2 + \frac{3}{4}.\frac{19}{16} + \frac{35}{128}}$.

(*b*) $\dfrac{(125)^2 + (125)36 + 180}{(125)^2 + 12(125) + 36}$.

9. The hypotenuse of a right-angled triangle is 3601 ft. One of the sides containing the right angle is 3599 ft. Find the length of the other side.

10. The kinetic energy of a body of mass m lbs. moving with a velocity of v ft. per second is $\frac{1}{2} mv^2$ ft. lbs. By what fraction is the kinetic energy diminished if the velocity of a train weighing 6 tons is reduced from 25 miles per hour to 20 miles per hour?

11. A stone when dropped from a height h takes t seconds to reach the ground; here h and t are connected by the formula $h = \frac{1}{2} gt^2$ where $g = 32$. Two stones are dropped from two heights and take respectively 32 seconds and 29 seconds. Find the difference of the heights from which they were dropped.

12. There are three rectangles 36″ × 26,″ 72″ × 8″ and 3″ by 3″. Show how these can be cut out and put together so as to form a square according to the formula $(a + b)^2 = a^2 + 2ab + b^2$.

13. Factorise the two expressions $2x^2 + 11x - 30$ and $2x^2 + 9x - 45$, and find that value of x which will make both the expressions zero.

14. Factorise $n^3 - 1$; and show that a number consisting of an even number of nines is always divisible by a number consisting of half as many nines as there are in the original number. Show also that this latter number increased by 2 is also a factor of the original number.

15. Draw a right-angled triangle. Measure the sides and show that the square on a side containing the right-angle is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side.

16. (*a*) Use the formula $(a+b)^2+(a-b)^2=2a^2+2b^2$ to simplify 1001^2+999^2.

(*b*) Show that $x^2+y^2=2\left(\frac{x+y}{2}\right)^2+2\left(\frac{x-y}{2}\right)^2$ and apply the formula to simplify $(101)^2+99^2$.

17. Use the formula $(a+b)(a-b)=a^2-b^2$ to express the product 98×78 as the difference of two squares.

18. Show that $(a+b)^3=a^3+3a^2b+3ab^2+b^3$ by multiplying by $(a+b)$ both sides of $(a+b)^2=a^2+2ab+b^2$. Hence (*a*) find the value of $(1001)^3$ without actual multiplication; (*b*) Simplify $(1{\cdot}009)^3$ with as little multiplication as possible.

19. Show that $(a-b)^3=a^3-3a^2b+3ab^2-b^3$ by multiplying by $(a-b)$ both sides of $(a-b)^2=a^2-2ab+b^2$. Hence find the value of $(999)^3$ without actual multiplication. (*b*) Simplify $({\cdot}998)^3$ with as little multiplication as possible.

20. 1 metre = 39·37 inches, find the number of cubic inches in a cubic metre with as little multiplication as possible.

21. Resolve x^2+6x+5 into factors. Put $x=10$ in the expression and the factors. To what arithmetical result does this lead?

22. Show that x^3+1 is divisible by $x+1$ and also x^3-1 by $x-1$. Take a number and its cube. Show that the next higher (or lower) number to the latter is always divisible by the next higher (or lower) number to the former.

23. Simplify, with as little actual multiplication, as possible:—

(1) $\dfrac{{\cdot}325\times{\cdot}325\times{\cdot}325+{\cdot}675\times{\cdot}675\times{\cdot}675}{{\cdot}325\times{\cdot}325-{\cdot}675\times{\cdot}325+{\cdot}675\times{\cdot}675}$.

(2) $\dfrac{({\cdot}001)^3-({\cdot}0001)^3}{{\cdot}001-{\cdot}001.}$

(3) $\dfrac{a^6-b^6}{a^4+a^2b^2+b^4}$ when $a={\cdot}876$ and $b={\cdot}875$.

24. (*a*) Show that $(n^2+1)^2-(n^2-1)^2=4n^2=(2n)^2$.

Give a series of consecutive values to n and show how a square number can be got by subtracting one square number from another.

(*b*) Show also that $(n^2+1)^2=(2n)^2+(n^2-1)^2$. By giving values to n show that a square number can be got by adding two square numbers.

(*c*) Hence show that n^2+1, n^2-1, and $2n$ represent the sides of a right-angled triangle.

25. Find the value of $(x-3)(x-5)$ when $x=1, 2, 3, 4, 5, 6, \ldots\ldots 10$ and find for what values of x the expression $(x-3)\times(x-5)$ becomes zero.

26. Similarly without giving different values to x, find for what particular values $(x-7)(x-8)$ will become zero.

27. Find the values of x which will make the expression x^2-4x+3 zero, or, in other words, solve the equation $x^2-4x+3=0$.

28. Solve the equations :—

(*a*) $3x^2+8x-3=0$.

(*b*) $6x^2-25x+14=0$.

29. Draw the graph of $y-x=0$ by forming a table of corresponding values of x and y, and also that of $y+x=0$ with the help of a similar table ; and show that the values of x and y in both the tables satisfy the equation $y^2-x^2=0$. What, do you think, is the reason for this ?

30. Draw the graph of the lines represented by the equation $y^2-4x^2=0$.

31. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ where $\pi = 3{\cdot}1416$. Find the difference in volume of two spheres whose diameters are 4·36 inches and 4·28 inches respectively, using the formula $a^3-b^3=(a-b)\{(a-b)^2+3ab\}$.

32. The sides of two cubes are 4·85 and 4·78 inches respectively ; find the difference in their contents using the formula given in Ex. **31.**

CHAPTER IV.

COMPOUND PROPORTION, VARIATION.

§ 29. Direct Variation. In part I, Chapter XX, we considered problems on proportion which involved only four quantities or two ratios. In this chapter we shall take up problems involving more than two ratios. Before we do that, however, it will be advantageous for us to acquaint ourselves with a new nomenclature and a new notation.

If a boy were to take a number of circles of diameters 1 in., 2 in., 3 in. &c., and measure their circumferences* as carefully as possible he would find the circumferences to be as shown in the following table :—

Dia.	1,	2,	3,	4,	5,	6 in.
Circ.	3·14,	6·28,	9·42,	12·57,	15·71,	18·85 in.

If any pair of corresponding values be examined, their ratio will be found to be 3·14 (correct to two places). Since the ratio is the same, whatever circle he may take, we see that the circumference of a circle bears a constant ratio to its diameter, *i.e.*, the circumference of a circle is *proportional* to its diameter or the circumference of a circle **varies** as its diameter.

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* The circumference of a circle may be measured in several ways. We may measure it by taking a thread round the circumference and finding the length of the thread by using the diagonal scale. Or, we may take a circular cylinder instead of a circle and wrap a paper round the cylinder, prick the paper where it overlaps, then unfold the paper and measure the distance between the prick-points using as before the diagonal scale.

[NOTE 1.—The ratio of the circumference to the diameter, as found from the above table, is 3·14. More careful experiments and calculation give the ratio to be 3·1416 correct to 4 places of decimals. The ratio may also be taken to be roughly $3\frac{1}{7}$. In other words, for all practical purposes, the circumference of a circle may be taken to be $3\frac{1}{7}$ times the diameter.

NOTE 2.—If the corresponding values of the diameter and the circumference be represented by points plotted on squared paper as explained in Chap. XIX, Part I, the points so obtained will be found to lie on a straight line passing through the origin, thereby showing that the ordinate, is proportional to the abscissae, *i.e.*, the circumference is proportional to the diameter.]

In general, A is said to **vary** directly as B if $A = k.B$ where k is some constant. k is called the **variation constant.**

A varies as B is symbolically written thus : – $A \propto B$.

We shall explain direct variation by a few examples.

Ex. 1. If 38 boys of a form pay fees amounting to Rs. 152, what amount of fees will be collected in a class of 29 boys belonging to the same form ?

We know that if the no. of boys be doubled, the amount of fees collected will be doubled, if trebled, the amount will be trebled, and so on, *i.e.*, the amount of fees is directly proportional to the number of boys. Using A for the amount of fees and N for the number of boys we have that A is proportional to N, *i.e.*, A varies directly as N, or $A = kN$ (where k is some constant) and is the same for all values of A and N.

We are given that A is 152 when N is 38.

$\therefore 152 = k.\ 38,$

$\therefore k = \frac{152}{38} = 4.$

$\therefore A = 4N.$

Now we are required to find the value of A when N = 29.

$\therefore A = 4 \times 29 = 116.$

The required amount of fees collected is Rs. 116.

The working may be slightly modified thus.

$A = kN$ for all values of A and N.

We are given that A is 152, when N is 38.

$\therefore 152 = k.38$ .....(1)

And if A be the required amount when N is 29

we have $A = k \times 29$ .....(2)

From (1) and (2), by division,

we have $$\frac{A}{152} = \frac{k \times 29}{k \times 38}$$

$$= \tfrac{29}{38}.$$

$$\therefore A = \tfrac{29}{38} \times 152 \text{ .....(3)}$$

$$= 116.$$

Equation (3) shows that A (the required amount of fees) is got by multiplying 152 (the original amount) by $\frac{29}{38}$, *i.e.*, the ratio of the number of boys in one case to the number of boys in the other case. On this account $\frac{29}{38}$ may be called the **multiplying ratio.**

Ex. 2. When weights are suspended from the free end of a wire fixed at one end, it is found that the extension (the increase in the length of wire) is proportional to, *i.e.*, *varies* as the weight suspended. Find the weight suspended when the extension is 0·15 cm., given that the extension is 0·12 cm. when the weight suspended is 8 Kg.

Method 1. Here W (the weight suspended in Kg.) varies as x (the extension in cm.)

i.e, $W = kx$,

given that $x = 0{\cdot}12$ when $W = 8$,

i.e., $8 = k.(0{\cdot}12)$

$$\therefore k = \frac{8}{\cdot 12}.$$

$$\therefore \text{ when } x = \cdot 15,\ W = \frac{8}{\cdot 12} \times \cdot 15$$

$$= 10.$$

Hence the weight suspended, when the extension is 0·15 cm., is 10 Kg.

Method 2. $W = k.\, x$ for all corresponding values of W and x,

also W is 8 when $x = \cdot 12$.

$\therefore\ 8 = k \times \cdot 12$.

If W be the required weight when $x = \cdot 15$

we have $W = k \times \cdot 15$.

By division

$$\frac{W}{8} = \frac{\cdot 15}{\cdot 12} \text{ or } W = \frac{\cdot 15}{\cdot 12} \times 8 = 10.$$

Note that the required weight is got by multiplying the given weight by $\frac{\cdot 15}{\cdot 12}$, *i.e.*, the ratio of the two given extensions.

[NOTE. In both these exercises, four quantities are involved three of which are given and the fourth is required to be found; and the required quantity is got by multiplying the given quantity of the same kind by multiplying by the ratio of the other two quantities. This ratio is easily determined by considering whether the required quantity is to be greater or less than the corresponding given quantity.

In EX. 1 worked above, when the no. of boys is decreased, the amount of fees collected is decreased, *i.e.*, the required

amount is less than the given amount, ∴ the multiplying ratio, $\frac{29}{38}$ has been less than unity.

In Ex. 2 when the weight suspended is increased, the extension is increased. ∴ the required weight is greater than the given weight. Hence the multiplying ratio is greater than unity. ∴ it is $\frac{\cdot 15}{\cdot 12}$ (which is greater than unity).

In actual practice such questions may be easily worked by finding the multiplying ratio.]

Exercise IV (a).

Work out the following according to the method of variation and constants and by the method of multiplying ratio.

1. A tree casts a shadow 16 ft. 6 in. long, while the shadow of a neighbouring tower is 6 inches. If the height of the tower be 91 ft. 6 in., what is the height of the tree?

2. If the carriage of 5 cwt. 7 lb. for 79¼ miles cost Rs. 118-2-0, what will it cost to have 10 cwt. 2qr. 14 lb. conveyed the same distance assuming the rate to be the same?

Work out all the questions in the chapter on Direct Proportion in Part I by this method.

§ 30. Inverse Variation.

Ex. 1 A garrison of 1800 men is provisioned for 12 weeks. How long will the provisions last (1) if 600 more men are introduced; (2) if 1800 more men are introduced; (3) if 600 men are withdrawn; (4) if 900 men are withdrawn.

It is clear that if the number of men is increased, the provisions will last for fewer days and if the number of men is diminished, they will last for more days. This is a case of inverse proportion. The answers, as worked by the

methods explained in Chapter XX, Part I, will be found to be

$$\frac{12 \times 1800}{1800 + 600}, \quad \frac{12 \times 1800}{1800 + 1800}, \quad \frac{12 \times 1800}{1800 - 600}, \quad \frac{12 \times 1800}{1800 - 900},$$

i.e., 9 wks., 6 wks, 18 wks. and 24 wks.

Let us arrange the corresponding values in a tabular form thus

No. of men in the garrison.	1800, 2400, 3600, 1200, 900.
No. of weeks for which provisions last	12, 9, 6, 18 24

Consider any pair of corresponding values, say 2400 and 9. Their product (which represents the total quantity of provisions) is 21600. The products of the other pairs of corresponding values, *viz.*, 1800×12, 3600×6, 1200×18 and 900×24 are each equal to 21600, a constant quantity.

$\therefore$ No. of men in the garrison $\times$ the No. of weeks for which the provisions last $= 21600$, which is constant.

$$\therefore \text{ the No. of weeks} = \frac{21600}{\text{the corresponding No. of men}}.$$

In this example of inverse proportion we see that when the number of men changes the number of weeks also changes and so changes that it is always got by dividing the constant 21600 by the corresponding number of men. In this case, the number of weeks is said to *vary inversely* as the number of men.

In general, A is said to *vary inversely* as B, when $A = \frac{k}{B}$ where k is some constant.

If a rectangle whose length and breadth are L and B be of constant area c, then $L \cdot B = c$ or $L = \frac{c}{B}$, *i.e.*, the length varies inversely as the breadth when the area is constant.

The following is another example on **Inverse Variation.**

Ex. 2. A ship has enough coal for 30 days using 125 tons a day; how long will it last if 150 tons a day are consumed?

We see clearly that the larger the quantity of coal consumed per day the smaller the number of days the supply will last and *vice versa*. Denoting the No. of days by D and the daily consumption by C, we have that D varies inversely as C,

i.e., $D = \frac{k}{C}$, (where k is the same for all corresponding values of D and C). But it is given that D = 30 when C is 125.

$$\therefore 30 = \frac{k}{125}.$$

$$\therefore k = 30 \times 125.$$

$$\therefore D = \frac{30 \times 125}{C}.$$

We are required to find the value of D when C = 150.

$$\therefore D = \frac{30 \times 125}{150} = 25.$$

∴ the required number of days = 25.

Or, thus:—

$D = \frac{k}{C}$ for all corresponding values of D and C.

Also D is given to be 30 when C is 125,

i.e., $= 30\frac{k}{125}$ ... ... (1)

If D′ be the required number of days when C is 150, we have $D' = \frac{k}{150}$ ... ... (2)

By division, $\frac{D'}{30} = \frac{125}{150}$ or $D' = \frac{125}{150} \times 30$.

Note. Here also the number of days required is got by multiplying the given number of days, *viz.*, 30 by $\frac{125}{150}$.

The note appended at the end of Art. 29 (page 64) regarding the multiplying ratio applies to the case of inverse variation also.

Exercise IV (b).

Work out the following according to the method of variation and constants and also according to the method of multiplying ratios.

1. An estate can be cut up into 31 plots each containing $2\frac{1}{4}$ acres. How many plots of $3\frac{7}{8}$ acres can be cut from it?

2. A hostel annually requires 180 bags of rice, each bag containing 40 measures of rice. How many bags will be wanted every year if bags containing 60 measures of rice are supplied, assuming the number of boarders to be the same?

Work out all the questions in the chapter on Inverse Proportion in Part I according to this method.

Compound Proportion.

§ **31.** So far we have had to work questions dealing with quantities of two kinds only, *e.g.*, in the first example worked under direct variation, the two quantities involved are the number of boys and the amount of fees collected, the one depending on the other. Such questions are called questions in **Simple Proportion.** In the following questions quantities of more than 2 kinds are involed, each quantity depending upon each of the other quantities and *vice versa.* Such questions are called questions in **Compound Proportion.**

Ex. 1. The area of a rectangle is 480 sq. ft. If the length is increased in the ratio of 4 : 5 and the breadth decreased in the ratio of 3 : 2, what will be the area of the new rectangle?

Let ABCD be the rectangle.

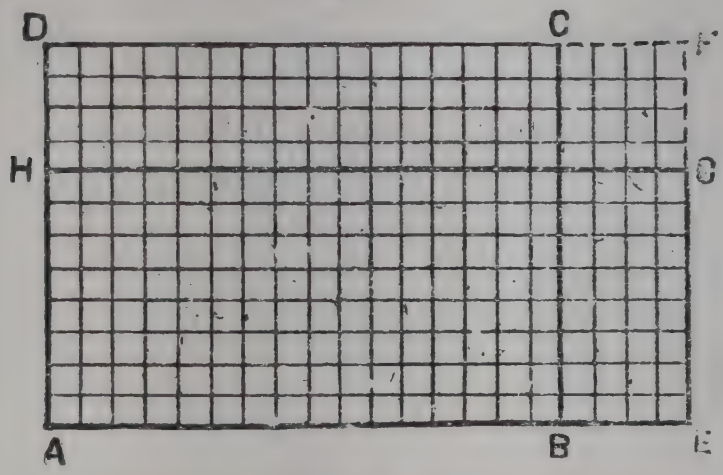


FIG. 33.

If its length be increased, in the ratio of 4 : 5, *i.e.*, from AB to AE (the breadth being unaltered), we pass from the rectangle ABCD to the rectangle AEFD ;

and rect. AEFD = rect. ABCD $\times \frac{5}{4}$.

If now the breadth be decreased in the ratio of 3 : 2 (keeping the new length unaltered), *i.e.*, from AD to AH, we pass from the rect. AEFD to the rect. AEGH ;

and rect. AEGH = rect. AEFD $\times \frac{2}{3}$

= rect. ABCD $\times \frac{5}{4} \times \frac{2}{3}$.

Hence rect. AEGH = 480 sq. ft. $\times \frac{5}{4} \times \frac{2}{3}$ = 400 sq. ft.

And the rect. AEGH has its length = $\frac{5}{4}$ the length of ABCD and its breadth = $\frac{2}{3}$ the breadth of ABCD.

Hence AEGH represents the new rectangle formed.

Ex. 2. If 2000 copies of a book of 360 pages require 96 reams of paper, how many copies of a book of 240 pages of the same size can be printed with 108 reams of paper?

Method 1 (By multiplying ratios).

Supposition.	2000 copies.	360 pages.	96 reams.
Demand.	?	240 pages.	108 reams.

As in the case of the rectangle in Ex. 1, let us consider the changes in the number of pages and in the number of reams to take place one after another.

(1) Consider the relation between the number of pages and the number of copies (supposing the number of reams to be constant). The number of copies varies inversely as the number of pages, *i.e.*, the number of copies is increased by diminishing the number of pages (supposing the number of reams to remain the same). ∴ the corresponding multiplying ratio is greater than unity and therefore equal to $\frac{360}{240}$.

(2) Consider the relation between the number of reams and the number of copies (supposing the number of pages to be constant). Evidently the number of copies varies directly as the number of reams, *i.e.*, the number of copies is increased by increasing the number of reams (supposing the number of pages per book is the same). ∴ the corresponding multiplying ratio is greater than unity and therefore equal to $\frac{108}{96}$.

$$\therefore \text{ the required number of copies} = 2000 \times \frac{360}{240} \times \frac{108}{96}$$

$$= 3375.$$

The student is recommended to compare each step in this solution with the corresponding step in the solution of Ex. 1 (*supra*) so as to clearly understand the reasoning.

Method 2. Questions in Compound Proportion can very often be reduced to questions in Simple Proportion.

The above question may be worked as follows:

Since 96 reams give 2000×360 pages,

108 reams ,, $\frac{2000 \times 360 \times 108}{96}$ pages.

But in the latter case 240 pages make 1 copy.

$\therefore \frac{2000 \times 360 \times 108}{96}$ pages make $\frac{1}{240} \times \frac{2000 \times 360 \times 108}{96}$

or 3375 copies.

* *Method* 3. (By variation).

We have seen that the number of copies varies inversely as the number of pages when the number of reams remains unaltered. $\therefore$ if c stands for the number of copies and p for the number of pages, we have $c = \frac{k}{p}$ where k is constant.. Now if p be changed to p' and consequently c to c_1 we have $c_1 = \frac{k}{p'}$. $\therefore$ By division,

$$\frac{c}{c_1} = \frac{p'}{p} \quad \ldots\ldots (1)$$

Now since the number of copies varies directly as the number of reams, say r, when the number of pages remains unaltered, say at p', we have $c_1 = k'r$ where k' is constant. Let now r be changed to r' and consequently c_1 to c' we have $c' = k'r'$.

$$\therefore \quad \frac{c_1}{c'} = \frac{r}{r'} \quad \ldots\ldots (2)$$

From (1) and (2) by multiplication, we have

$$\frac{c}{c'} = \frac{p'}{p} \cdot \frac{r}{r'}. \quad \therefore c' = c \cdot \frac{p}{p'} \cdot \frac{r'}{r} \ldots \quad \ldots\ldots (3)$$

Putting in the values of c, p, p', r and r' we have the required number of copies equal to $2000 \cdot \frac{360}{240} \cdot \frac{108}{96} = 3375$.

(*N.B.*—The result (3) could be arrived at, if we assume the relation $c = \frac{k \cdot r}{p}$ ..... (4)

Where c, r and p respe ctively denote the corresponding values of the number of copies, the number of reams and the number of pages and k is a constant: for, then if r and p change to r' and p' and consequently c to c' we have $c' = k \cdot \frac{r'}{p'}$, ... (5)

and from (4) and (5) by division we have

$$\frac{c'}{c} = \frac{r'}{p'} / \frac{r}{p} = \frac{p}{p'} \cdot \frac{r'}{r}. \quad \therefore c' = c \cdot \frac{p}{p'} \cdot \frac{r'}{r}.$$

The assumption (4) is also legitimate; for, c varies directly as r when p is constant, and varies inversely as p when r is constant. And generally if A varies directly as each of a, b,..... and inversely, as each of x, y.....when all the others are constant, we may assume $A = k \cdot \frac{ab.....}{xy.....}$

Exercise IV (c).

1. If the railway fare for 8 persons for a journey of 384 miles is Rs. 32, what must be paid for 120 miles for 48 persons (the cost of a ticket being proportional to the distance)?

2. A stable company spends Rs. 226-0-8 per week for feeding 25 horses, what will be the cost of keeping 12 horses for 72 days?

3. How much will it cost to send 2 maunds 30 seers a distance of 42 miles, if it costs 1 anna to send 1 maund 1 mile (the cost being supposed to be proportional to the distance and the weight)?

4. If the wages of 20 coolies working 8 hours a day be Rs. 60 in a week of 6 working days, how much will 35 coolies working 6 hrs. a day earn in 3 weeks (all days of the week being working days) assuming that they all get the same amount per hour?

5. If 20 men can reap 11 acres in 7 hrs., how many acres will 14 men reap in 16½ hours?

6. In a Hindu marriage which lasted for 4 days the cost of candles required for illumination purposes amounted to Rs. 26-4-0, and each candle could burn for 6 hours; find what the cost of candles would be in a marriage lasting for 6 days, supposing each candle to burn for 7 hours (the cost of a candle being the same in each case).

7. If 25 lbs. of cotton make 120 yds. of cloth 1 yd. wide, how much cloth 1½ yds. wide ought 15 lbs. to make?

8. If the cost of cloth 28 in. wide required to make a coat for an adult is Rs. 21¼ at Rs. 4-4-0 per yard, find the cost of cloth

32 in. wide required to make a coat for the same adult at Rs. 3-8-0 per yard.

9. By what method will you work the preceding question if a fraction of a yard cannot be sold ?

10. In the celebration of the Brahmotsavam festival lasting for 10 days, a temple hired a number of Washington lights which were placed at intervals of 100 yds. along the route of the procession at 12 as. per light. The charges amounted to Rs. 750. What did the charges come to in another year assuming the celebrations lasted for 12 days, that the lights were placed at intervals of 80 yds. and hired at 14 as. each ?

11. If 15 pumps working 6 hours a day can raise 945 tons of water in 7 days, how many pumps working 8 hrs. a day will be required to raise 7560 tons of water in 14 days ?

12. If 120 men can build a wall 25 ft. high in 15 days, how many men will be required to build a similar wall to a height of 30 ft. in 10 days ?

13. When the price of butter is 12 as. per measure, the ghee required per month for a family of 4 adults and 2 children costs Rs. 15 ; what will be the cost of ghee per month for a family of 10 adults and 8 children, when the price of butter is Re. 1 per measure, assuming that a child takes only half as much as an adult, and that an adult of the latter family consumes twice as much ghee as an adult of the former family ?

14. If the cost of printing a book of 240 leaves with 20 lines on each page be Rs. 200, find that of printing a book with 253 leaves with 15 lines on each page (the length of a line in each case being the same).

15. The garrison of a besieged town consists of 500 men with provisions that will last for 10 weeks, allowing 9 oz. per day for each man. If the garrison be reinforced with 125 more men, to what must the daily allowance be reduced in order that the provisions may last for 12 weeks ?

16. A contractor undertakes to dig a canal 12 miles long in 350 days and employs 45 men ; he finds that in 200 days he has completed $4\frac{1}{2}$ miles ; how many additional men must be employed to get the undertaking finished in time ?

17. A ship of 2500 horse-power consumes 168 tons of coal in 4 days; how many pounds of coal does it consume per horse-power per hour?

18. A motor-car, moving uniformly, covers a kilometre in 30 sec. If 2 km. = 1·25 miles, express the speed in miles per hour.

19. If with a pace of 2 ft. 6 in. I walk a mile in 25 minutes, how long will it take me to walk the same distance if I quicken my steps in the ratio of 4 to 5 but decrease the length of my pace to 2 ft. 4 in.?

20. A train which does 40 miles in 60 min. describes a certain distance in $3\frac{1}{2}$ hrs. How long will another train which does 35 miles in 55 minutes take to describe the same distance if they both travel uniformly?

**Variation.

§ **32.** We have considered hitherto only simple cases of variation. But in Physics and in practical examples you may have one quantity varying directly or inversely as the square of another, the cube of another, &c., *e.g.* (1) In the case of a freely falling body, the space described varies as the square of the time from starting. (2) The area of a circle varies as the square of the radius. (3) The volume of a sphere varies as the cube of the radius, &c.

If y varies as the square of x we may express the relation as $y = kx^2$ where k is a constant. (This may be established by the method of §§ 29, 30.)

If y varies as the cube of x we have $y = kx^3$ where k is a constant, &c.

Ex. 1. The kinetic energy of a body is proportional to the square of its velocity. If the kinetic energy of a body moving with a velocity of 50 ft. per second is 625 foot-pounds, what must be the kinetic energy of the body when it moves with a velocity of 60 ft. per second?

The kinetic energy, *viz.*, K.E $= k.v^2$

It is given to be 625 when v is 50.

$\therefore\ 625 = k.(50)^2$

When v is 60, K.E. $= k.(60)^2$.

$$\therefore\ \frac{\text{KE}}{625} = \frac{(60)^2}{(50)^2}.$$

$\therefore$ K.E. $= 625 \times \frac{36}{25} = 900.$

The required kinetic energy $=$ 900 foot-pounds.

Ex. 2. The resistance in ohms of a conductor (of uniform section) is directly proportional to its length in miles and inversely proportional to the area of its cross section given in sq. inches. Copper wire, one-twelfth of an inch in diameter, has a resistance of 8 ohms per mile: What is the resistance of 3 miles of copper wire the diameter of which is $\frac{1}{36}$ in. ?

We have $\text{R} = k\frac{l}{\text{A}}$ where R is the resistance in ohms., l the length in miles and A the area in sq. in.

But A varies as d^2 (diameter)^2.

$$\therefore\ \text{R} = \frac{kl}{d^2}.$$

We are given that R is 8 when l is 1 and the diameter is $\frac{1}{12}$.

$$\therefore\ 8 = k.\frac{1}{(\frac{1}{12})^2}.$$

If x be the required resistance in ohms.

$$\therefore \text{ then } x = k.\frac{3}{(\frac{1}{36})^2}$$

$$\therefore \text{ by division, } \frac{x}{8} = \frac{3}{1}.\frac{(\frac{1}{12})^2}{(\frac{1}{36})^2}.$$

$$\therefore\ \frac{x}{8} = 3.\frac{36^2}{12^2}.$$

$$\therefore\ x = 24.3^2 = 216.$$

$\therefore$ the required resistance is 216 ohms.

§ **33.** In all the preceding examples we have assumed that y varies directly or inversely as some power of x. The relation between x and y may be more complicated. We shall take up the case in which x and y are connected by a relation of the form

$$y = ax + b,$$

where a and b are constants, *i.e.*, remain the same for all values of x and y. This presents an important type of examples which may be solved graphically also. A student who has carefully worked through the exercise given under Art. 167, Part I, will see that if $y = ax + b$, the corresponding graph is a straight line.

Ex. 1. The expenses of a hostel are partly constant and partly vary directly as the number of boarders. If the expenses come to Rs. 286 when the number of boarders is 22 and to Rs. 412 when the number of boarders is 34, find the expenses when the number of boarders is 28. Draw a graph so as to enable you to find the expenses corresponding to any number of boarders.

Let the expenses in rupees be E, and the number of boarders N. Then by the question

$$E = kN + c.$$

Where c represents the constant expenditure in Rupees.

It is given that E is 286 when N is 22 and is 412 when N is 34.

$\therefore$ we have $286 = k.22 + c$ ... ... (1)

and $412 = k.34 + c$ ... ... (2)

Subtracting, $126 = k.12$.

$$\therefore k = 10\tfrac{1}{2}.$$

Substituting in (1), we have

$$286 = 10\tfrac{1}{2} \times 22 + c$$

$$\therefore c = 55.$$

∴ we have, in general,

$$E = 10\tfrac{1}{2} \times N + 55 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

and when N = 28 we see that

$$E = 10\tfrac{1}{2} \times 28 + 55$$
$$= 349.$$

∴ the expenses when the number of boarders is 28 amount to Rs. 349.

Graphically.—In this question if the expenses in rupees be y and the number of boarders x, the relation connecting y and x is given by $y = kx + c$ where k and c are constants. The graph is, therefore, a straight line. When x is 22, y is given to be 286, *i.e.*, (22, 286) is a point on the graph. Similarly (34, 412) is a point on the graph. And the required graph is therefore the straight line passing through these two points. [That the required graph is a straight line can be verified by plotting sets of corresponding values of N and E got from the equation (3) given above]. Now the expenses corresponding to 28 boarders is got by reading

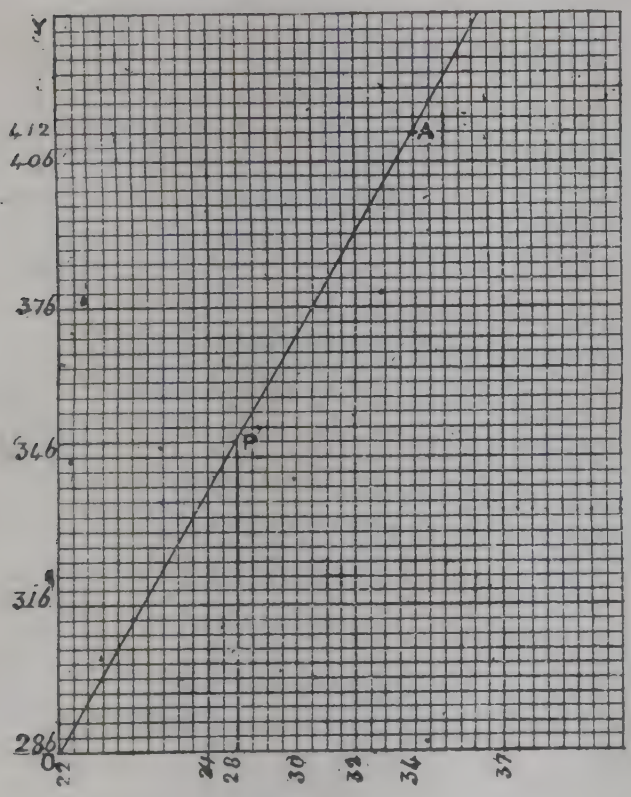


FIG. 34.

the ordinate corresponding to the abscissa 28. From the figure it is found to be 349.

[In the annexed figure (22,286) is taken as the origin and two divisions on the x axis to represent one boarder and one division on the y axis to represent Rs. 3. If we want the expenses corresponding to any number of boarders below 22 or the constant part of the expenses we must choose other suitable axes or produce these backwards and downwards.]

Exercise IV (d).

1. When the price of gram is 11 measures per rupee the cost of gram for 24 ponies for 31 days is Rs. 182-2-0. Assuming that 5 horses require as much as 8 ponies, find the cost of gram for 25 horses for 6 weeks, when the price is $10\frac{1}{2}$ measures per rupee. (*Mad. Matr. 1897*).

2. A contractor undertook to finish a certain piece of work in 150 days. He employed 20 men, 30 women and 75 children, but at the end of 60 days finding that only one-fourth of the work was done, he dismissed all the women and 50 of the children and employed more men. The work was then finished 5 days before the stipulated time. Assuming that 3 men could do as much as 5 women, and 2 women as much as three children, find how many additional men were employed. (*Mad. Matr. 1896*).

3. When iron was £2-14-2*d*. per ton and the rupee at 1*s*. $7\frac{1}{2}d$., the cost of laying a railway with iron rails weighing 50 lbs. per yard was Rs. 2,78,250. Find what will be the cost of relaying it with steel rails weighing 75lbs. per yd, when the price of steel is £3 17*s*. 6*d*. per ton and the rupee is at 1*s*. $1\frac{1}{4}d$. (*Mad. Matr. 1895*).

4. Zinc sheets 24 ft. by 4 ft. 8 in. are reckoned to weigh 36 to the ton. How many sheets 6 ft. by 4 ft. 3 in. of twice the thickness would you expect to go to the ton? Answer to the nearest integer.

5. If a cubic yard of gravel is needed for a path 3 ft. wide and 8 yds. long and costs Re. 1-2, find the cost of covering with gravel a path $6\frac{1}{2}$ ft. wide and 150 ft. long. What length of path 4 ft. wide can be covered with gravel for Rs. 35-8-0 ?

6. If the cost of metalling a road 10 miles 2 fur. 55 yds. long and 9 ft. broad with metal costing Rs. 2-8-1 per cubic yd. be Rs. 24,165-11-5, find what will be the cost of metalling a road 22 miles 1 fur. 132 yds. long and 15 ft. broad with metal costing Rs. 2-3-3 per cubic yd., the depth of the metal in the latter case being half as great again as the depth in the former case.

7. The area of a circle is proportional to the square of its radius. If the area of a circle of radius 16 in. is 800 sq. in., what is the area of a circle of radius 9 in. ?

8. The area of a circle is proportional to the square of its radius. If the area of a circle of 16 in. radius is 800 sq. in., what is the radius when the area is 200 sq. in. ?

9. The volume of a given quantity of gas is inversely proportional to the pressure. If the volume is 145 c.c. when the pressure is 72·5 cm. of mercury, what is its volume when the pressure is 70 cm. of mercury ?

10. The weight of a sphere of silver is 6lbs., what would be the weight of a sphere of gold, of thrice the diameter, gold being 2 times as heavy as silver ? (The volume of a sphere varies as the cube of its diameter).

11. The flow of water from a 3-in. pipe is 40 gallons per minute. If in a 2-in. pipe water flows as fast, what is the flow of water per minute ?

12. Given that the weight of a rope is proportional to the square of its circumference, fill in the following table :—

Circumference.	2	4	5	6.
Weight (per fathom).	—	—	22·5	—.

Draw a graph and from the graph verify your results.

13. Show that the values of x and y given below are connected by a relation of the form $y = ax+b$ where a and b are constants. Find also the values of a and b.

$$x = 0,\ 9,\ 12,\ 15.$$
$$y = 5,\ 59,\ 77,\ 95.$$

14. The intensity of illumination is directly proportional to the illuminating power of the source and inversely proportional to the square of its distance from the screen. If a candle is placed at a distance of 1 ft. from a cardboard screen and a lamp of 9 candlepower is placed at a distance of 12 ft. on the other side, compare the intensity of illumination on the two sides of the screen.

15. Assuming that the area of a circle varies as the square of the circumference and that the area of a circle of 6 in. circumference is 2·8665 sq. in., find a formula giving the area of any circle when its circumference is known.

16. A ball rolls down a plane and it is found that in 3 seconds it rolls down 6 ft. Assuming that the distance rolled down is proportional to the square of the time, find how long it will take to roll down 24 ft.? Find an equation connecting d the distance rolled down and t the time.

17. A metal plate 2·5 metres long, 1·82 m. broad, 5 cm. thick, weighs 432·5 Kg. How much less would it weigh, if it had been made 6 cm. less broad and 3 mm. less thick?

18. A cistern in the form of a cube whose edge is 5 ft. holds 7808 lbs. of water. How much water will a cistern of the same shape hold whose edge is 3 ft. 4 inches?

19. A boat with 1360 sq. ft. of sail-area travels 25·8 knots in 8 hours when the wind gives a pressure of ·125 lbs. per square foot. How far will it go in $4\frac{2}{3}$ hrs. with a sail-area of 7650 sq. ft. and a wind-pressure of ·5 lb. per square foot, assuming the speed to be proportional to the area of the sail and the wind-pressure?

20. The angle of inclination to the vertical of a bicycle moving round a circular track on horizontal ground varies (approximately) as the square of the velocity and inversely as the circumference of the track. If the inclination is 6^0, when the velocity is 20 miles an hour and the circumference of the track $\frac{1}{3}$ mile, what is it when the

velocity is 2 miles an hour less and the circumference 440 ft. shorter?

21. Assuming that the areas of circles are proportional to the squares of their radii, compare the areas traced out by the short and long hands of a clock in 40 minutes, the long hand being 2·5 times as long as the short hand.

22. The diameters of four spheres are to one another as $3\frac{3}{8} : 4\frac{1}{2} : 5\frac{5}{8} : 6\frac{3}{4}$ and the volume of a sphere varies as the cube of the diameter. Prove that the greatest of these four spheres is equal in volume to the other three put together.

23. The weights of a penny, a half-penny and a farthing are in the ratio of 10 : 6 : 3 and their diameters are as 6 : 5 : 4. Find the ratio of their thicknesses.

24. An elastic string of length ·477 m. is fastened at its upper end and is stretched to a length of ·504m. by a suspended weight of 18 grammes. Find to what length a piece of the same string, a yd. long is stretched by $\frac{1}{2}$ oz. weight (given that a gramme = ·0022 lb. and that the ratio of the increase of length to the original length is always proportional to the suspended weight).

25. Out of a circular disc of metal, 35 equal circular holes are punched. The weight of the metal thus punched out is to the weight of the perforated disc as 45 : 67. Compare the diameters of the disc and of the holes. (The area of a circle varies as the square of the diameter).

26. The income of a touring officer is partly constant and partly variable, the latter part being proportional to the number of days he is away from the headquarters. In a certain month he tours for 20 days and draws Rs. 390. In another month he tours for 23 days and draws Rs. 402. Find what his income is when he is out for only 10 days. Give also a graphical solution. Also find from the graph what the constant part of his income is.

27. The charge for luggage for a given journey is $a(x - b)$ annas, where x maunds is the weight and a and b are constants, (b maunds is the weight allowed free of charge.) If 50 mds. of luggage was charged Rs. 12-6-0 and 100 mds. Rs. 24-14-0, how

much can be carried free and what is the charge for each extra maund ?

28. The charge for press messages is as follows :—Ordinary—8 as. for the first 48 words and one anna for every additional 6 words. Express—one rupee for the first 48 words and two annas for every additional 6 words.

Find a formula in each case connecting the total charge in annas (say, t) with the number of words (say, w). From the formula find what the charge will be for an express message containing 500 words. Draw a graph illustrating your answer.

29. A telegraph company charges Rs. 3-5 for a message of 500 words but charges Rs. 3-15 for another message of 600 words. Presuming each of these charges to consist of (1) a charge for a fixed number of the first few words (2) a charge proportional to the excess number of words above that fixed number, find what the cost for a message of 750 words will be. Work out the question graphically.

30. The weight of water that a bucket holds varies as the depth and as the square of the diameter of the top. A bucket 12 in. deep and 15 in. across the top will just hold 84·375 lbs. of water. Find a formula giving the weight of water in pounds in terms of the depth and the diameter of the top.

31. A specific gravity bottle weighs 14·72 grammes when empty, 39·74 grammes when filled with water and 44·85 grammes when filled with a solution of common salt. What is the specific gravity of the solution ?

32. It costs Rs. 30 to gild a sphere which weighs 20 cwt. What will it cost to gild a sphere of the same material which weighs 160 cwt.? [The superficial area of a sphere varies as the square of the radius and the weight as the cube of the radius.]

33. The square of the time that a planet takes to revolve round the sun varies as the cube of its distance from the sun. If the mean distance of Jupiter from the sun in terms of the earth's distance is 5·20, how long will Jupiter take to revolve round the sun if the earth takes 1 year ?

34. A body dropped from a height h ft. reaches the ground in t seconds. From the following table of values of h and t examine (1) whether h varies as t or as t^2. Draw a graph.

h	16	64	144	256	400	576	784
t	1	2	3	4	5	6	7

35. The pressure P at a depth Z below the surface of a liquid is given in pounds per square foot. From the following table of values of P and Z examine (1) whether P varies as Z or Z^2.

P	640·6	960·9	1601·5	1921·8	3203
Z	10	15	25	30	50

Draw a graph and from your figure find the pressure at a depth of 40 ft. Also find at what depth the pressure is 1000 pounds per square foot.

36. The volume of a quantity of gas is v cubic centimetres when the barometer stands at p centimetres. From the following table of values of p and v, test if pv is constant and also draw a graph showing the relation of p to v.

p =	76	72	70	64	60	56
v =	612	646	664·5	726·8	775·2	803·6

37. The length of a zinc rod at temperatures $t°$ and 0° may be denoted by l_t and l_0. From the following table show that $l_t = l_0 (1 + t \times \cdot 000029)$.

l_t	100·15	100·29	100·44	100·58	100·73
t	50°	100°	150°	200°	250°

$l_0 = 100$ cm.

38. If v_t and v_0 denote the volume of a quantity of gas at t^0 and 0^0 respectively, show from the following table that

$$v_t = v_0\left(1 + \frac{t}{273}\right) \text{ very nearly.}$$

v_t	320·22	340·44	350·55	371·42	400
t	20°	40°	50°	65°	91°

$v_0 = 300$ c.c.

39. Experiments on the expansion of benzene gave the following results. If v, the volume be supposed given by the formula $v = a + bt$ when t is the temperature, determine the values of the constants a and b.

Temperature ...	0°	20°	40°	60°	80°
Volume ...	1	1·0241	1·0500	1·0776	1·1070

40. Three successive measurements for finding the focal length of a convex lens were made with the following results :—

Experiment 1 ... ... $p = 40$ cm., $p' = 66$ cm.
„ 2 ... ... $p = 45$ cm., $p' = 60$ cm.
„ 3 ... ... $p = 62$ cm., $p' = 43$ cm.

Calculate the focal length f according to each measurement from the formula $\frac{1}{f} = \frac{1}{p} - \frac{1}{p'}$; find also the mean value of the focal length.

41. The greatest load that a beam supported at its ends will carry is directly proportional to the breadth, and to the square of the thickness and inversely proportional to the length. If a beam 20 in. long, 2 in. broad and $\frac{3}{4}$ in. thick will support 40 lbs; how much will one 50 in. long 4 in. broad and 2·5 in. thick support ?

42. The distances and first class railway fares from Madras to the undermentioned stations are as follow, the distances being reckoned in miles and the fares in rupees and annas :

	Distance in miles.	Fares. RS. AS.
Nellore	110	10 5
Padugudadu	113	10 10
Kodavaluru	117	11 0
Talamarchi	120	11 4
Allum Road	127	11 15
Bitragunta	131	12 5
Kavali	142	13 5
Ongole	182	17 1

From these data draw on squared paper a diagram showing the relation between the distance travelled from Madras and the railway fare. Examine whether the railway fare is proportional to the distance travelled assuming distances to be given to the nearest mile and 1 anna to be the smallest coin used for payment of railway fares.

43. An isosceles triangle has its equal sides each equal to 6 cm. while the vertical angle increases from 0 to 90°. Fill up the following table by drawing figures corresponding to the several values of the vertical angle and measuring the corresponding bases. Show by a graph the changes in the length of the base, taking 1 inch horizontally to represent 15° and 1 inch vertically to represent 1 cm. From the graph read off to the nearest degree the size of the vertical angle when the base is 8 cm. and to the nearest millimetre the length of the base when the vertical angle is 80°.

Vertical angle ...	15°	30°	45°	60°	75°	90°
Base in cm. ...	...	...	...	...	...	...

44. Construct a triangle ABC on a base of 6 cm. having the base angles B and C equal to 25° and 35° respectively. Measure *b* and *c*. Show that *a*, *b* and *c* are nearly proportional to the sines of the angles A, B, C, given that

Sine 120° = ·8660,
Sine 35° = ·5736,
Sine 25° = ·4226.

45. The following table gives the measurements of two quantities x and y which are known to be connected by a formula of the form $y = a + bx$; but the measured values are slightly wrong. Plot the values of x and y on squared paper; find the most likely values of a and b.

x	1	1·5	2·5	3·4	5·2	6·0
y	·225	·270	·338	·405	·54	·6

46. The monthly expenses of a College Hostel are partly constant and partly variable, the latter being proportional to the number of boarders and the standard of prices of articles. In a certain month when there are 100 boarders, the expenses amount to Rs. 800, In the next month when there are only 90 boarders and when the prices have increased in the ratio of 7 to 9, the expenses amount to Rs. 910. Find what the expenses would be in a month in which the number of boarders was only 70 and the articles became cheaper than in the second month in the ratio 9 : 8.

47. s is the sum of two quantities one of which varies as t and the other as t^2; $s = 1344$ when $t = 8$; and $s = 2000$ when $t = 10$. Find s when $t = 12$ and give the formula connecting s and t.

48. y is the sum of two quantities one of which varies directly as x and the other varies inversely as x, If $y = 3\frac{1}{2}$ when $x = 6$ and $y = 2\frac{3}{4}$ when $x = 4$. Find y when $x = 12$ and give the formula connecting y and x.

CHAPTER V.

SIMPLE EQUATIONS AND PROBLEMS.

§ 34. Equations involving fractions.

In this chapter we shall work some equations which require the clearing of fractions before any transformations can be made.

Ex. 1. $\frac{1}{3}(x-2)-\frac{1}{5}(x-4)=\frac{1}{6}(5x-2)$.

The L.C.M. of the denominators 3, 5, 6 is 30. ∴ if we multiply by 30 the equation will be free from fractions.

Multiplying every term of the equation by 30, we have

$$10(x-2)-6(x-4)=5(5x-2).$$

Removing the brackets

$$10x-20-6x+24=25x-10.$$

Transposing $10x-6x-25x=-10+20-24$.

$$\therefore \quad -21x=-14$$

$$\therefore \quad x=\frac{14}{21}=\frac{2}{3}.$$

Verification :—

Left-hand side

$$=\frac{1}{3}\left(\frac{2}{3}-2\right)-\frac{1}{5}\left(\frac{2}{3}-4\right)=\frac{1}{3}\left(-\frac{4}{3}\right)$$

$$+\frac{1}{5}\times\frac{10}{3}=-\frac{4}{9}+\frac{2}{3}=\frac{2}{9}.$$

The right-hand side

$$=\frac{1}{6}\left(5\times\frac{2}{3}-2\right)=\frac{1}{6}\left(3\frac{1}{3}-2\right)=\frac{1}{6}\left(\frac{4}{3}\right)=\frac{2}{9}.$$

∴ the root we have got is correct.

Ex. 2. Solve :—

$$\frac{1}{3}\left(5-\frac{x}{3}\right)+\frac{2}{9}\left(\frac{7}{8}x+5\right)=2\frac{3}{4}+\frac{1}{2}\left(\frac{3x}{2}+4\right).$$

Removing brackets, we get

$$\frac{5}{3}-\frac{x}{9}+\frac{7}{36}x+\frac{10}{9}=\frac{11}{4}+\frac{3x}{4}+2.$$

The L.C.M. of the denominators 3, 9, 36 and 4 is 36.

Multiplying every term of the equation by 36, we get

$$60-4x+7x+40=99+27x+72.$$

$$\therefore\quad -4x+7x-27x=99+72-60-40.$$

$$\therefore\quad -24x=71.$$

$$\therefore\quad x=-\frac{71}{24}=-2\frac{23}{24}.$$

The student is required to verify the correctness of the solution.

Ex. 3. $(x+3)(x+5)+(2x+3)(3x+1)$
$=(x+2)(3x+1)+(4x+5)(x-6).$

Expanding, we have,

$$x^2+8x+15+6x^2+11x+3$$
$$=3x^2+7x+2+4x^2-19x-30$$

$7x^2$ appears on both sides and may be removed.

Removing $7x^2$ and transposing, we have

$$8x+11x-7x+19x$$
$$=2-30-15-3.$$

$$\therefore\quad 31x=-46$$

$$\therefore\quad x=\frac{-46}{31}=-1\frac{15}{31}.$$

Ex. 4. $\dfrac{x}{8} - \dfrac{(2x+1)(3x+1)}{7}$

$= \dfrac{2}{5} - \dfrac{(6x+3)(5x+7)}{35}.$

Clear off fractions by multiplying every term by 280.

$35x - 40(2x+1)(3x+1)$
$= 112 - 8(6x+3)(5x+7).$

$\therefore\ 35x - 40(6x^2 + 5x + 1)$
$= 112 - 8(30x^2 + 57x + 21).$

$\therefore\ 35x - 240x^2 - 200x - 40$
$= 112 - 240x^2 - 456x - 168.$

The term $-240x^2$ is common to both sides and may be removed.

$\therefore\ 35x - 200x + 456x$
$= 112 - 168 + 40.$

$\therefore\ 291x = -16.$

$\therefore\ x = -\dfrac{16}{291}.$

Exercise V (a).

Solve and verify the roots :—

1. $12(x+5) = 16x + 40.$

2. $8(x-5) + 6(x-2) = 8(2x-9).$

3. $3(x-4) + 17(x-8) = 9(3x-2) + 5(4x-3) - 132.$

4. $3(2x+4) - 8(6x+7) = 10(4x+3) - 17(3x+8).$

5. $2 - \{3 - (5-x)\} = 6(x-7).$

6. $3x - 6\{8 - (x-6) + 4x\} - 28 = 35.$

7. $4[15 - 3\{x - 2(x-3)\}] - (6x - 18) = 0.$

8. $\dfrac{6x+5}{4} = \dfrac{3x-1}{8}.$

9. $\dfrac{4x-11}{7} + \dfrac{8x-9}{3} = 0.$

10. $3x - \frac{1}{3}(7x+9) = 18.$

11. $\frac{x}{4} - \frac{6x+7}{6} = \frac{3x-5}{8}$.

12. $\frac{4x-1}{6} + \frac{7}{12} = \frac{x}{8} + \frac{3x+5}{9}$.

13. $\frac{7x-5}{10} + \frac{x-2}{4} = \frac{4x-3}{5} - \frac{6+x}{15}$.

14. $\frac{3x-2}{3} + \frac{4x-3}{4} = \frac{5x-4}{6} - \frac{2x+3}{7}$.

15. $7 - \frac{x-2}{3} - \frac{x-4}{5} = \frac{4-x}{6}$.

16. $\frac{1}{3}(1-7x) - \frac{1}{8}(4-3x) + \frac{17}{48} = 0$.

17. $\frac{1}{2}(x+3) + \frac{1}{3}(x+4) = \frac{1}{5}(x+5) + 16$.

18. $\frac{1-2x}{3} + \frac{2-3x}{4} + \frac{8-6x}{7} = 0$.

19. $\frac{9}{5}(8-x) + \frac{6}{7}(4-3x) + \frac{1}{4}(x-3) = 8$.

20. $\cdot 5x - 8 = \cdot 65x + 3$.

21. $\cdot 2(x-1) + \cdot 3(5x-3) = \cdot 8(6x-7)$.

22. $\frac{\cdot 25 - x}{3} + \frac{\cdot 68 + 3x}{4} = \frac{8\cdot 8x}{1\cdot 25}$.

23. $(x+1)(x-2) = x(x+6) - 4$.

24. $2(x-1)(x-3) = (x-2)(2x-5) + 6$.

25. $23 - x(8-x) = (x-5)^2$.

26. $(3x-1)(2x+5) + (4x+8)(x-7) = (5x+3)(2x-7) + 30$.

27. $(5x+2)(x+3) - (4x-1)(3x-2) = 12 - (7x+3)(x-2)$.

28. $8(x-1)(x-3) - 7(x-3)(x-4) = (x-1)^2 + 4(3x+2)$.

29. $\frac{(7x+2)(4x-3)}{4} - \frac{(8x-3)(4x-2)}{16} = \frac{(6x-1)(5x+3)}{6}$.

30. $4 + \frac{(3x-1)(4x-2)}{16} - \frac{x^2}{4} = \frac{x^2-2}{2}$.

31. $\frac{x(3x+1)}{2} = \frac{(2x+3)(3x+4)}{4} - 2\frac{1}{2}$.

32. $\frac{1}{3}(x-\frac{1}{3}) + \frac{1}{4}(x+\frac{2}{3}) = \frac{1}{5}(x+1)$.

33. $\frac{2}{5}(\frac{7x}{8} - \frac{1}{9}) = \frac{6}{7}(\frac{14x}{25} = \frac{1}{78})$.

34. Show that the following equations are identities :—

(i) $(3x+5)(4x+3) - (6x+2)(2x+5) = 5 - 5x$.

(ii) $\frac{3}{5}(x+4) - \frac{2}{7}(x-9) = \frac{11}{35}x + 4\frac{34}{35}$.

35. Find the value of x which makes the two expressions $(3x - 5)(8x + 4)$ and $(4x + 3)(6x - 1)$ equal.

§ **35.** Equations are useful in Mathematics because by means of such equations many problems can be solved. Before such problems are attempted the student should acquire considerable facility in stating the conditions of the problem in algebraical language so as to form an equation. Some exercises in such symbolic expression have been provided in Part I and here are some advanced examples.

(1) The digits of a number beginning from the left are x, y, z. What is the number? Find the value of the number formed by reversing the digits.

z is the digit in the units' place ; y in the tens' place ; and x in the hundreds' place.

$\therefore$ the number $= 100x + 10y + z$.

If the digits are reversed the digits beginning from the left will be z, y, x. Hence the value of the new number is $100z + 10y + x$.

(2) A has p rupees, B has q annas ; A hands r rupees to B and finds that he then has twice as much as B. Express this fact algebraically.

A has p rupees and hands over r rupees.

$\therefore$ A has left with him $(p - r)$ rupees

or $16(p - r)$ annas.

B has q annas and gets r rupees from A.

$\therefore$ he ultimately has q as. $+ r$ rupees

or $(q + 16r)$ annas.

By the question A has ultimately twice as much as B.

$\therefore \quad 16(p - r) = 2(q + 16r).$

Note.—Both sides of the equation must be expressed in the same denomination.

The student is recommended to work the above question again expressing the sums in rupees instead of in annas and thus verify the equation we have arrived at.

Exercise V (b).

1. The product of a and b is equal to 4 times the excess of c over d. Express this algebraically.

2. If p is divided by q the quotient is greater than the sum of r and s by 5. Express this in algebraical symbols.

3. A man who is m years old has a son whose age is s years. 3 years hence the father's age will be twice that of his son. Express this in symbols.

4. A has x mangoes and B has y times as many ; C has z mangoes which is 100 more than what A and B together have. Express this algebraically.

5. A walks at a miles an hour for h hours ; B walks at b miles an hour for k hours and finds that he has walked l miles less than A. Express this fact in the form of an equation.

6. The length of a rectangular tank is 4 times its depth and its breadth is twice its depth. If it holds a tons of water and a cubic foot of water weighs b lbs., express the depth in terms of a and b.

7. Three spheres whose diameters are a, b and c inches are melted down and recast into a single sphere. If the radius of the new sphere is d in., find a relation between a, b, c and d.

8. Two circular discs whose diameters are x and y inches are melted down and recast into a single circular disc. of the same thickness as each of the former. If z in. is the diameter of the new disc, find a relation between x, y and z, assuming that the areas of circles are proportional to the squares of the diameters.

9. The area of a triangle whose base is b units and whose height is h units is equal in area to a square whose side is s units. Establish a relation betwen b, h and s.

10. The sides of a right-angled triangle are a and b ft. and the hypotenuse is less than the sum of the sides by 4 ft. Find the relation between a and b.

11. The sum of the three sides of a triangle is p; the three sides are in the ratio of $x : y : z$. Find the lengths of the sides separately. Find an additional relation between x, y, and z, if the triangle is right-angled (x corresponds to the greatest of the sides).

12. The diagonals of a rhombus are d_1 and d_2. If A is the area of the rhombus, establish a relation between A, d_1 and d_2. [The angle between the diagonals of a rhombus is a right angle].

13. The lengths of the minute-hand and the hour-hand of a watch are m and h inches. If s represents the difference between the area of the dial and that generated by the motion of the hour-hand, establish a relation between m, s and h, assuming that the area of a circle is πr^2 where π is a constant and the area generated by the minute-hand is the whole dial.

14. If when the length of a room l ft. long and b ft. broad is increased by x ft. and the breadth diminished by y ft. the area remains unaltered, find the relation between l, b, x and y.

15. The dimensions of a rectangular room are l, b and h ft., while those of another room are l', b' and h' ft. The area of the walls of the former room is twice the area of those of the latter room; express this fact algebraically.

16. A rectangular field 4 times as long as it is broad contains s acres. Express the perimeter in yards.

17. The external dimensions of a closed box are a ft., b ft. and c ft. and the material is d in. thick. If it holds h ounces of water and 1 cubic ft. of water weighs 1000 ounces, establish a relation between a, b, c and h.

18. A, B and C agree to divide a property worth p rupees among themselves in the ratio of $x : y : z$ and it is given that A's share is to be equal to B's and C's put together. Express this algebraically.

19. The difference of a quantity and its reciprocal is equal to the square of the quantity. Express this symbolically.

§ 36. In solving problems by means of equations we first denote the unknown quantity by x and express the

conditions of the problem in symbolical language. In every problem it is quite possible to get two expressions which are numerically equal. The equation so framed may be solved.

Ex. 1. The sum of two numbers is 60. 4 times the smaller exceeds three times the greater by 16. Find the numbers.

Let x be the greater of the numbers; then the smaller number is $60 - x$.

4 times the smaller $= 4(60 - x)$.

3 times the greater $= 3x$.

$\therefore$ the excess of 4 times the smaller over 3 times the greater $= 4(60 - x) - 3x$ and this equals 16, by the question.

$\therefore \quad 4(60 - x) - 3x = 16.$

$\therefore \quad 240 - 4x - 3x = 16.$

$\therefore \quad -7x = 16 - 240 = -224.$

$\therefore \quad x = \dfrac{-224}{-7} = 32,$

i.e., the greater number $= 32$.

$\therefore$ the smaller number $= 60 - 32$ or 28

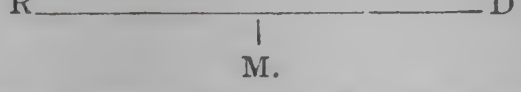
NOTE (1). The answer to a problem should not be given in the form $x = 32$ but x should be replaced in the answer by what it stands for.

NOTE (2) The solution should always be tested to see whether it satisfies the conditions of the problem.

Here $32 + 28$ is 60; $4 \times 28 - 3 \times 32$ is 16 and hence our solution is correct.

Ex. 2. AB is a road 360 miles long. Two motor-cars start from A and B at the same time and travel in opposite directions along the road at the rate of 35 miles per hour and 37 miles per hour. When and where will they meet?

Sol. 1. Let the cars meet after x hours from the time of starting.



Let the place of meeting is M. RM be the distance travelled in x hours by the car starting from R.

$\therefore$ RM $= x \times 35$ miles, and similarly DM $= x \times 37$ miles.

$\therefore$ RM + DM, *i.e.*, RD $= (35x + 37x)$ miles

$\therefore$ by the question, RD $= 360$ miles $\therefore$ $35x + 37x = 360$.

$\therefore$ $x = \frac{360}{72} = 5$.

$\therefore$ RM $= 5 \times 35$ or 175 miles.

$\therefore$ the cars meet 5 hours after starting and the place is 175 miles distant from R.

NOTE. The cars move in opposite directions and they cover between them $35 + 37$ or 72 miles in every hour. The whole distance described being 360 miles, the time that lapses before they meet is $\frac{360}{72}$ or 5 hours. This is exactly what we have got.

Sol. 2. Let the place of meeting be x miles from R, *i.e.*, let RM be x miles.

Then DM $= (360 - x)$ miles.

The time taken by the car leaving R to travel the distance RM at the rate of 35 miles per hour $= \frac{x}{35}$.

Similarly the time taken by the othre car to travel DM is

$$\frac{360 - x}{37}.$$

and these times are equal.

$\therefore$ $\frac{x}{35} = \frac{360 - x}{37}$.

$\therefore$ $37x = 35(360 - x)$.

$\therefore$ $72x = 35 \times 360$.

$\therefore$ $x = \frac{35 \times 360}{72} = 175$,

i.e., the place of meeting is 175 miles from R and the time taken is $\frac{175}{35}$, or 5 hours.

Sol. 3. *Graphical.* Let O be the origin.

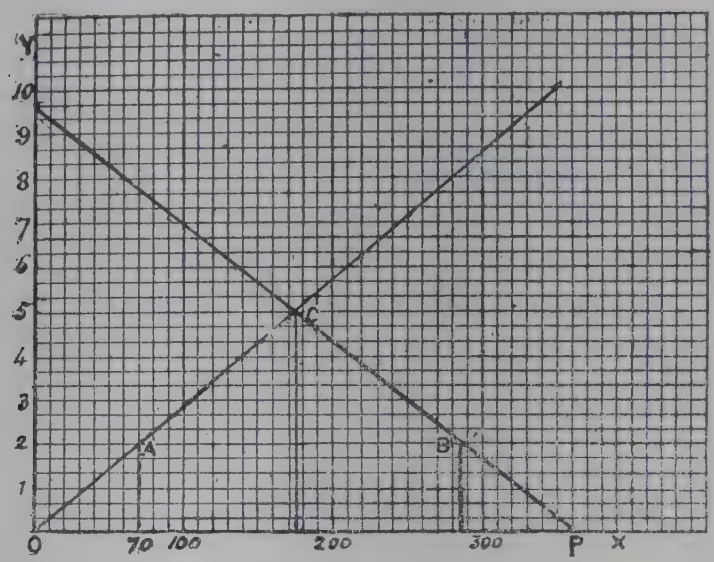


FIG. 35.

Let one division on the x axis represent 10 miles, and let 3 divisions on the y axis represent an hour. Mark a point P on the x axis 36 divisions from O, so that OP may represent 360 miles. As the first car moves at 35 miles an hour, if we plot (35, 1)* and join it with O we get the graph representing the motion of the first car. The other car starts in the opposite direction from P and moves at 37 miles an hour. Hence if we plot a point whose abscissa is 37 to the left of P and whose ordinate is 1 and join it with P we get the graph of the other car.† These two graphs meet at C whose abscissa is 175 miles and whose ordinate is 5 hours.

N.B.—The graph tells us something more than what is actually wanted. It tells us when the cars reach their respective destinations. The graph passing through the origin meets the parallel through 360 at a point whose ordinate is a little less than 10 hours 20 minutes. Similarly, the graph passing through P meets the y axis at a point whose ordinate is a little less than 9 h. 40 minutes.

* For convenience (70, 2) is marked in this figure and joined to O.

† In this figure a point whose abscissa is 74 to the left of P and whose ordinate is 2 is marked and joined to P.

Ex. 3. (*a*) The hour-hand and the minute-hand of a watch are found to be together between 3 and 4. What time is indicated?

(*b*) What is the time indicated when they are at right angles, the hour-hand being between 3 and 4?

(*a*) Let the hands be together at x minntes past 3. At 3, the minute-hand is at 12 and the hour-hand at 3. Therefore, in the time in which the minute-hand has moved through x minute-divisions, the hour-hand has moved through $(x - 15)$ minute-divisions. But the rates of the minute-hand and the hour-hand are 60 minute-divisions per hour and 5 minute-divisions per hour.

And since distances travelled in the same time are in the ratio of the rates

$$x : x - 15 = 60 : 5$$
$$= 12 : 1$$
$$x = 12(x - 15).$$
$$11x = 180$$
$$\therefore \quad x = 16\tfrac{4}{11}.$$

$\therefore$ the time indicated is $16\frac{4}{11}$ ms. past three.

(*b*) Let the two hands be at right angles at x minutes past 3. The minute-hand moves x minute-divisions from 12. The hour-hand is 15 minute-divisions behind; hence it has moved through $(x-15-15)$ divisions (for it started from the fig. III, *i.e.*, it was 15 minute-divisions in advance.)

$$\therefore \quad x : (x - 15 - 15) :: 12 : 1$$
$$x : (x - 30) :: 12 : 1$$
$$12x - 360 = x$$
$$11x = 360$$
$$x = 32\tfrac{8}{11}.$$

$\therefore$ the time indicatd is $32\frac{8}{11}$ ms. past three.

NOTE.—Another position when they are at right angles is 3 o'clock itself.

Ex. 4. A cyclist A riding at 8 miles an hour set out at 7 A.M. and after every complete hour of riding he rests for 5 minutes. Another cyclist B riding at 10 miles an hour sets out at 7-30 A.M. in pursuit of A. B also rests for 5 minutes after each complete hour of riding. When and where does B overtake A?

Graphical Solution.

Let O be the origin and represent 7 A.M. on the x axis. Let one division on the x axis denote $2\frac{1}{2}$ minutes and one division on the y axis represent half a mile. At 7 A.M., A is at the starting point, *i.e.*, at O. Plot A the point (8 A.M., 8).

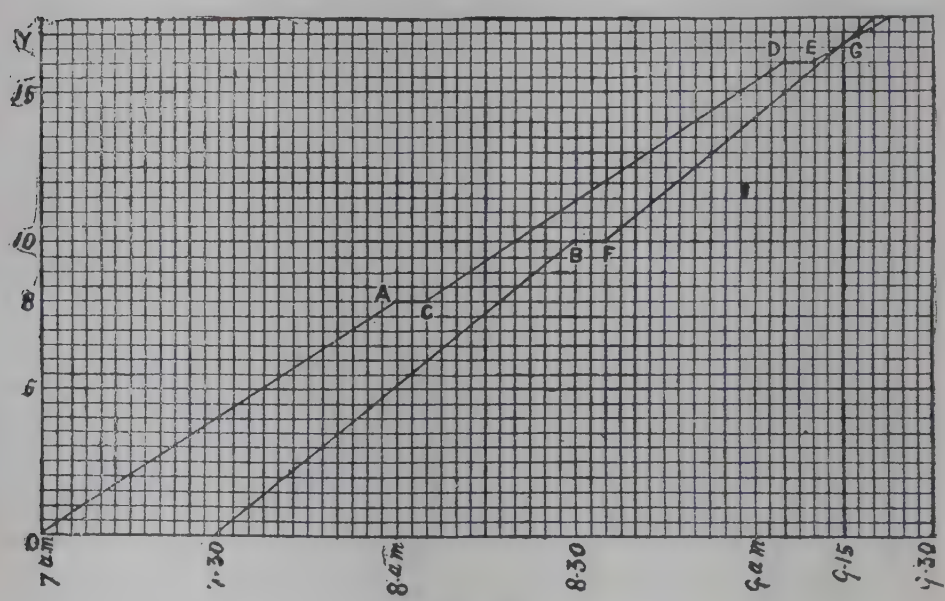


FIG. 36.

Join it to O. OA is A's graph for the first hour. Then A takes rest for 5 minutes. His graph during that time is a level line AC (there is no ascent because he is at the same mile). Since his subsequent rate is the same as at first a line CD parallel to OA will be his graph. At 9-5 A.M. he again rests and his graph again is a level line indicated by DE in the

figure. From E draw a line parallel to OA. Thus OACDEG is the graph of A's motion.

To get B's graph. Plot 7-30 on the x axis and plot (8-30, 10). Join these two points. After 8-30 his graph is a level line. Then proceed as in the case of the first graph. You find that the two graphs intersect at a point whose abscissa denotes the time 9-15 A.M. and the distance is a little less than 17 miles.

Algebraical Solution.

Let B overtake A x hours after 7 A.M. From the graph it is clear A rests twice, *i.e.*, for 10 minutes or $\frac{1}{6}$ hour.

$\therefore$ A rides in all for $(x - \frac{1}{6})$ hours.

The distance ridden by A $= (x - \frac{1}{6}) \times 8$ miles.

B starts at 7-30 A.M. and rests only once, *i.e.*, for 5 min.

$\therefore$ B rides in all for $(x - \frac{1}{2} - \frac{1}{12})$ hours and the distance ridden by B $= (x - \frac{7}{12}) \times 10$ miles.

When B overtakes A both have described the sme distance,

$$\therefore \quad 8(x - \tfrac{1}{6}) = (x - \tfrac{7}{12})\,10$$

$$8x - \tfrac{8}{6} = 10x - \tfrac{70}{12}.$$

$$\therefore \quad 2x = \tfrac{85}{6} - \tfrac{8}{6} = \tfrac{27}{6}.$$

$$x = 2\tfrac{1}{4}.$$

i.e., B overtakes A after $2\frac{1}{4}$ hours, *i.e.*, at 9-15 A.M.

The distance described $= 8(x - \frac{1}{6}) = 16\frac{2}{3}$ miles.

N.B.—In this question the graph helps the algebraic solution, though tentatively we can find the number of times each rests before B overtakes A.

Ex. 5. Two kinds of ghee are worth Re. 1-5 and Rs. 2-4 a viss respectively. In what proportion should they be mixed so that the mixture may be worth Re. 1-14 per viss?

Let x viss of the second sort be mixed with every viss of the first sort in order that the mixture may be worth Re. 1-14 a viss. Now a viss of the first sort costs Re. 1-5 or 21 as. and x viss of the second sort cost x (Rs. 2-4) or $36x$ as. Thus the total cost of this mixture is $(36x + 21)$ as. But the mixture contains $(x + 1)$ viss and costs Re. 1-14 or 30 as. a viss; and its cost, therefore, is $30(1 + x)$ as. Therefore, we have $36x + 21 = 30(1 + x)$

or $6x = 9$. $\therefore x = \frac{3}{2}$.

Thus for every viss of the first sort if we take $1\frac{1}{2}$ viss of the second sort and mix them we get a mixture of the required worth.

(*Verification.*—A viss of the first sort costs 21 as., $1\frac{1}{2}$ viss of the second sort costs (36 + 18) or 54 as., *i.e.*, $2\frac{1}{2}$ viss of the mixture cost 75 as. $\therefore$ one viss costs 30 as. or Re. 1-14).

Arithmetical Solution.

For every viss of the first sort taken, there is a fall of (Re. 1-14 — Re. 1-5) or 9 as. in the price of the mixture; and for every viss of the second sort mixed up, there is a rise of (Rs. 2-4 — Re. 1-14) or 6 as. in the price of the mixture. Thus to compensate the fall of 9 as. caused by taking one viss of the first sort, the quantity of the second sort (a viss of which produces a rise of 6 as.) which we must take is $\frac{9}{6}$ or $1\frac{1}{2}$ viss.

Graphical Solution.

Let each division on OX (Fig. 37) represent 2 palams and each division on OY denote 1 anna.

Plot the point P (1 viss, 21 as.); Q (1 viss, 36 as.); and

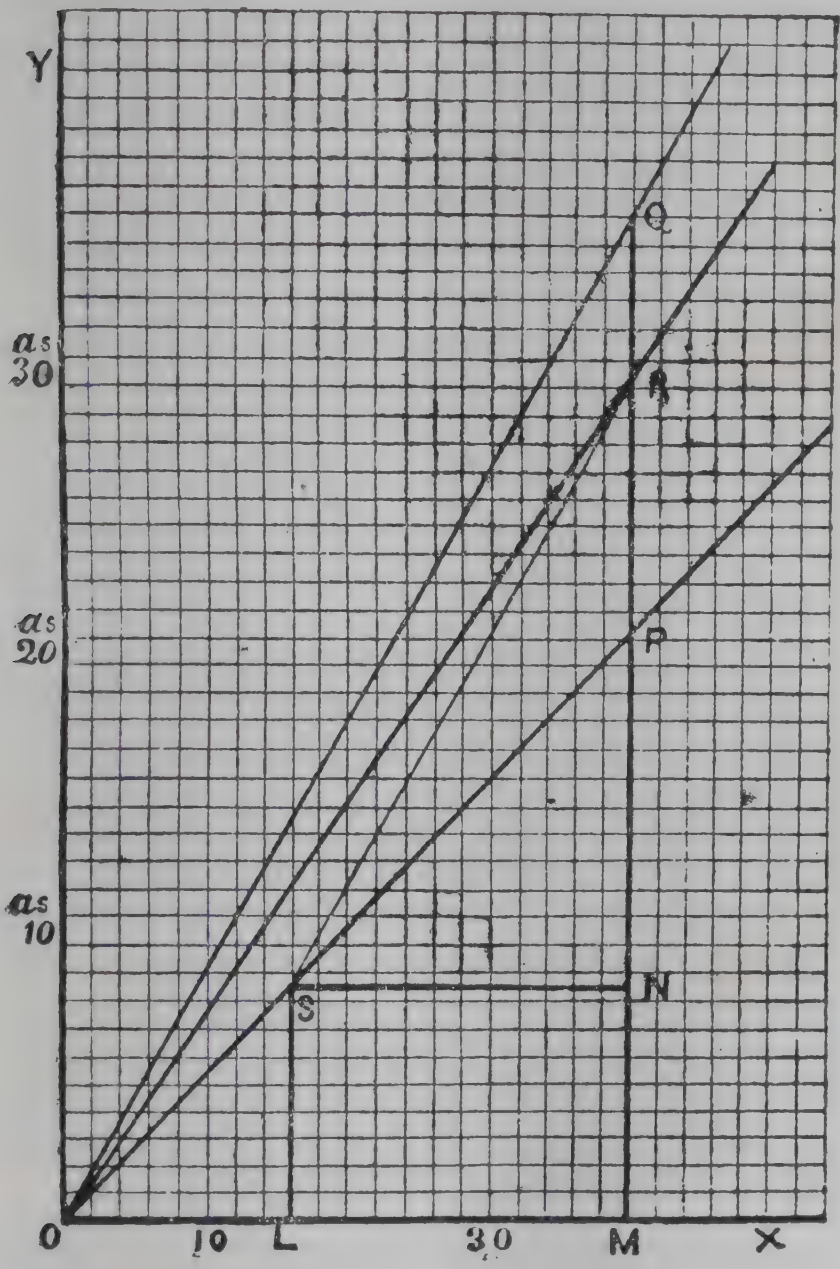


FIG. 37.

R (1 viss, 30 as.) Join OP, OQ and OR. These are respec-

tively the graphs of the first kind, the second kind and the mixture. The ordinate RM which represents the price of a viss of the mixture, *viz.*, Re. 1-14 which is the sum of two prices corresponding to the quantities of the first and second sort contained in a viss of the mixture, is therefore, the sum of two ordinates relating to each of the graphs OP and OQ. Through R draw a parallel to OQ to meet OP at S. Draw SL, perpendicular to OX, and SN perpendicular to RM. Now, ordinate RM=sum of the ordinates SL and RN. S is a point on OP and SL corresponds to the abscissa OL which represents the corresponding quantity of the first sort. Again SR has the same slope as OQ (OQ being parallel to SR) and is, therefore, the graph of the second sort. The ordinate RN (of graph RS) corresponds to the abscissa SN or LM which, therefore, represents the corresponding quantity of the second sort. Thus OL or 16 palams of the first sort mixed with LM or 24 palams of the second gives 40 palams or a viss of the mixture worth (SL + RN) or RM, *i.e.*, 30 as. So the ratio in which the two kinds of ghee are to be mixed is 16 : 24 or 2 : 3.

Ex. 6. A certain passenger train travels at the rate of 15 miles per hour, and a goods train, which is half as long again as the passenger train, at 9 miles per hour. When the two are travelling in the same direction, it takes 50 seconds for the passenger train to clear the goods train. Find the lengths of the trains and find how long it takes the two trains to pass each other when they are travelling in opposite directions.

Part I. It is required to find the length of each train. Let x yds. be the length of the passenger train. Then $1\frac{1}{2}\,x$ yds. is the length of the goods train. In the initial position, *i.e.* at the instant of E_1 overtaking G_2, E_2 and G_1 are

separated by a distance $= x + 1\frac{1}{2}x$ or $2\frac{1}{2}x$ yds. The passenger train is said to clear the goods train when G_1 comes up to E_2 and the time taken is 50 seconds.

15 miles per hour = 22 ft. per second
and 9 miles ... $= \frac{66}{5}$ ft. ,,

In 50 seconds G_1 travels 50×22 ft.

Again.....E_2..... $50 \times \frac{66}{5}$ ft.

$\therefore \quad 50 \times 22 - 50 \times \frac{66}{5} = 2\frac{1}{2}x \times 3.$

$\therefore \quad 50(22 - \frac{66}{5}) = \frac{15}{2}x.$

$\therefore \quad 50 \times \frac{44}{5} = \frac{15}{2}x.$

$x = 50 \times \frac{44}{5} \times \frac{2}{15} = \frac{176}{3},$

i.e., the length of the passenger train $= \frac{176}{3}$ or $58\frac{2}{3}$ yds.

.....goods train = 88 yds.

Part II. When the trains travel in opposite directions, in the initial position E_1 and E_2 meet, G_1 and G_2 are separated by $l + 1\frac{1}{2}l$ yds. (where l yds. is the length of the passenger train) and in the final position G_1 and G_2 meet. Let the trains pass each other in x seconds.

In x seconds, G_1 travels $x \times 22$ ft.

In.....G_2..... $x \times \frac{66}{5}$ ft.

$\therefore \quad x \times 22 + x \times \frac{66}{5} = (\frac{176}{3} + 88)3.$

$\therefore \quad x \times \frac{176}{5} = 440.$

$\therefore \quad x = \frac{440 \times 5}{176} = \frac{25}{2} = 12\frac{1}{2},$

i.e., they pass each other in $12\frac{1}{2}$ seconds.

NOTE. In both the cases, the distance involved in passing is the same, *viz.*, the sum of the lengths of the trains. When they are moving in the same direction it is the *difference* of the velocities that describes this distance, and when they are moving in opposite directions it is the *sum* of the velocities that describes this distance. Since the time taken in describing any distance is inversely proportional to the velocity, the time in the first case is to the time in the second case as the sum of the velocities is to their difference. In this case the sum is 24 miles per hour and the difference 6 miles per hour. ∴ the time in the second case is to the time in the first case as $6 : 24 = 1 : 4$, 50 seconds being the time in the first case, $12\frac{1}{2}$ seconds is the time in the second case, *i.e.*, in passing each other.

Exercise V (c).

1. A number is multiplied by 4 and 34 is subtracted from the result. When the remainder is divided by, 10 the quotient is 11. Find the number.

2. The sum of 6 times a certain number and 90 is equal to the difference between 26 times the number and 10. Find the number.

3. $\frac{1}{4}$ of a number increased by 18 is equal to $\frac{1}{6}$ of the difference between twice that number and two. What is the number?

4. The sum of twice the least of three consecutive numbers and the largest falls short of four times the middle number by six. Find the numbers.

5. Divide Rs. 168 between A, B and C so that B's share may exceed twice A's, and C's may fall short of thrice the same, by as. 4.

6. Three partners A, B and C contribute a sum of Rs. 3180. A contributes half as much again as B less Rs. 60, while C contributes twice B's share and Rs. 120 more. Find the amount contributed by A.

7. A bag contains Rs. 96 in 8 anna, 4 anna, and 2 anna, pieces The number of 2-anna pieces is half as much again as the number of 8-anna pieces and the number of 4-anna pieces is half as much as the number of 8-anna and 2-anna pieces put together. Find the number of 2-anna pieces the bag contains.

8. One fourth of a pole and 3 inches are painted black, one-third the remainder and 4 inches more, red, $\frac{1}{6}$ the remainder and 3 inches more, white, leaving 7 ft. to be painted green. Find the length of the pole.

9. A man buys 200 mangoes and some oranges for Rs. 14, when they respectively cost Rs. 6 and Rs. 4 per hundred. How many oranges does he buy? Illustrate your answer graphically.

10. Ghee costs Re. 1 4 as. a viss more than oil. They are mixed in the ratio of 9 : 7 and the mixure costs Re. 1-7-3 per viss. What is the cost of 1 viss of pure ghee?

11. Rs. 2000 is to be invested in two banks, one of which pays 5 per cent. interest and the other $4\frac{1}{2}$ per cent. interest. If the interest realised in one year from the invested sum is to be Rs. 96, find the amount to be invested at 5 %.

12. A labourer was hired on condition that for every day he worked he should receive 6 as. and every day he was absent from his work he should forfeit 3 as. If, in the month of January, he received Rs. 3-3-0, how many days was he absent from his work?

13. I buy a horse and a saddle for Rs. 500. By selling the horse at a gain of 15 per cent. and the saddle at a loss of 20 per cent. I gain 8 per cent. on the whole. What price did I pay for the horse?

14. A man pays income-tax at the rate of 6*d.* in the pound. If his gross income were increased by £10 and the tax raised to 8*d.* in the pound his net income would be unchanged; what is his gross income?

15. A and B begin to play a game with equal sums of money, and B loses Rs. 241 A now has Rs. 8 more than twice what B has. With what sums of money did they begin the game?

16. A is 36 years older than B now and 7 years hence would be our times as old. What is B's age now?

17. There is a number of two digits whose difference is 2 and if the number be diminished by $\frac{3}{2}$ times the sum of its digits, the order of the digits will be reversed. Find the number.

18. A grocer mixes 30 manuds of one kind of rice which costs Rs. 4 a maund with a certain quantity of another kind of rice which costs Rs. 3-8 a maund and finds that the mixture costs Rs. 3-13 a maund. Find how many maunds of the second kind he has mixed.

19. The distance between two points P and Q is 42 miles. A and B start from P and Q respectively at the same time and walk towards one another at the rate of $2\frac{1}{2}$ and $3\frac{1}{2}$ miles an hour respectively. At what distance from P will they meet? Illustrate graphically.

20. If in the previous question, A starts half an hour after B, when and where will they meet? Illustrate graphically.

21. If in question 19, A and B reach Q and P and return to P and Q, when and where will they meet? Illustrate graphically.

22. A horse goes from X to Y in 3 hours and walking 3 miles less per hour it takes 4 hours more for the return journey. At what speed did the horse go in the first case?

23. A train starts from Bangalore to Mysore at 7 A.M. travelling at 12 miles an hour. Another leaves Mysore for Bangalore at 9-40 A.M, and travels at 15 miles an hour. Find where and when they met (the distance between Bangalore and Mysore is 86 miles.)

24. A man walks up a mountain at the rate of 4 miles an hour and down again by a route 6 miles longer at the rate of $6\frac{1}{2}$ miles an hour. If he is out 4 hours altogether how far has he walked uphill?

25. The distance between two railway stations, P and Q, is 200 miles. A train leaves P at 8 A.M. and meets with an accident at 9-30 A.M. which detains it for one hour. Another train which leaves Q at 9 A.M. at the rate of 40 miles per hour meets this train at 11-4 A.M. What is the speed of the first train?

26. At what time between 2 and 3 o'clock will the hour and minute hands of a watch be at a distance of 20 minutes from one another?

27. When will the hour and minute hands of a watch be (1) in the same straight line. (2) at right angles to one another, between 7 and 8 ?

28. At what time between 3 and 4 will the hands of a watch be equally inclined to the vertical ?

29. A train travelling at the rate of 36 miles an hour overtakes a person walking on a road parallel to the railway and passes him in 6 seconds ; it also meets another person walking at the same rate as the other but in the opposite direction and passes him in 4 seconds. Find the length of the train.

30. A man started for a walk when the hands of his watch were coincident between 4 and 5 o'clock ; when he finished, the hands were again coincident between 5 and 6 o'clock. What was the time when he started and how long did he walk ?

31. A and B start at noon from two towns 45 miles apart, A's rate of walking being twice B's. If they walk 6 hours before they meet, find their rates of walking.

32. The sum of the fifth and sixth parts of a certain number exceeds the difference between its fourth and seventh parts by 545. Find the number.

33. A broker bought 150 shares of a Company which cost him Rs. 3,750 ; he reserved some, and sold the remainder for Rs. 3,488 gaining Rs. 7 a share on the cost price. How many shares did he reserve ?

34. A man gives a boy 40 yds. start in 200 yds. and loses the race by 20 yards. What would have been a fair start to give ?

35. AB is a straight line 5·6 cm. long. P is a point in AB such that PA : PB :: 4 : 3. Find the lengths of the segments PA and PB (1) when P is in the line AB ; (2) P is in the line produced.

36. At two stations A and B on a line of railway, the prices of coal are 15*s.* and 17½*s.* per ton respectively. If the distance between A and B be 100 miles and the cost of carriage of coal be ¼*d.* per ton per mile, find the distance from A of a station on the line at which the price of coal, had it come either from A or B, would be the same.

37. Two trains 88 yds. and 66 yds. long travel at 20 and 30 miles an hour on parallel rails in opposite directions. In how many seconds will they pass each other ?

38. Two trains 88 yds. and 66 yds. long run on parallel rails in opposite directions and pass each other in 5 seconds. If one of them travels at 36 miles an hour, find the rate of the other.

39. One of the sides of a right-angled triangle is 15″ and the hypotenuse is greater than the other side by 10″. Find the length of the other side.

40. The side of a square is increased by 100 yds. and the corresponding increase in area is 10 acres. Find the side of the square.

41. The difference between the squares of two consecutive numbers is 101. Find the numbers.

42. A piece of lead (sp. g. 11·54) weighs 137 grams in water. What is its weight in vacuum ?

43. A body of sp. g. 5 weighs 25 grams in vacuum ; what will the body weigh when immersed in water ?

44. Find the position of the centre of mass of two spheres of brass 1 inch and 3 inches in diameter placed at a distance 5 inches apart (the distance between the centres), according to the formula $m_1 d_1 = m_2 d_2$, where d_1 and d_2 are respectively the distances from the centre of mass of the masses m_1 and m_2. [Masses of spheres are proportional to the cubes of their diameters.]

45. The acceleration f of two masses m_1 and m_2 attached to a string passing over a pulley and starting from rest is given by the formula $f = \frac{g(m_1 - m_2)}{m_1 + m_2}$. If $f = 20$ cm. per second per second and $g = 980$ cm. per second per second and $m_1 = 5$ gr., find m_2.

CHAPTER VI.

APPROXIMATIONS, ERRORS AND CONTRACTED METHODS.*

§ 37. No measurements taken in practical life are ever quite exact. Ordinarily, when we say that the length of a line measured by our foot-rule is 3·4 inches, we do not mean that it is exactly 3·4 in. but we only mean that it is nearer to 3·4 in. than to 3·3 in. or to 3·5 in. If asked to find its length more accurately, we use the diagonal scale and give the length as, say, 3·42 in. Even here we only mean that so far as the eye can see, the length is 3·42 ; or rather, the length is nearer to 3·42 than to 3·41 or to 3·43.

Generally the extent to which we can trust our measurements depends upon the nicety of the instruments used and the care with which the measurements are taken. For example, in using a scale, an untrained man will be able to read only correct to the smallest division marked on the scale, whereas a trained eye will be able to read 'correct to a tenth of that division' In any case, all measurements when expressed in figures will be correct only to a certain number of figures, though the degree of accuracy may differ with different individuals and different instruments. And it will be shown in this chapter that in the calculation of results based on such measurements, it will be meaningless and even misleading if we carry our operations, such as multiplication and division, beyond a certain number of

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* Before beginning this chapter the student is recommended to read §§ 9, 48, 50, 66 and 80 of Part I.

places, and that, consequently our work of multiplication, &c., may be contracted to a great extent.

§ **38. Error.** We have already learnt that when we speak of a number as being 3·14 when expressed "correct to two decimal places," we mean that it is nearer to 3·14 than to 3·13 or to 3·15, *i.e.*, that it is greater than 3·135 and less than 3·145. Therefore in taking the number as 3·14, there is a possible **error** of ·005 in excess or in defect, *i.e.*, the real number may be as much as ·005 greater than or as much as ·005 less than 3·14. This may be symbolically expressed thus:

3·14 correct to 2 decimal places = 3·14 ± ·005.

The **limits of error** in taking 3·14 for the number are thus ± ·005. The actual error may have any value lying between these extreme values.

Ex. A sum of money when expressed in rupees correct to two places of decimals is Rs. 4·92. Write down the greatest and least possible value of the given sum.

The sum may lie between Rs. 4·915 and Rs. 4·925.

Rs. 4·915 = Rs. 4 + ·915 × 16 as.
= Rs. 4 + 14·640 as.
= Rs. 4-14 as. + ·64 × 12 pies
= Rs. 4-14-7·68 pies.

Rs. 4·925 = Rs. 4 + ·925 × 16 as.
= Rs. 4 + 14·800 as.
= Rs. 4-14 as. + ·8 × 12 pies.
= Rs 4-14- 9·6 pies

The greatest value = Rs. 4-14-9·6
„ least „ = Rs. 4-14-7·68.

As we do not generally go beyond pies, the greatest value correct to a pie is Rs. 4-14-10 and the least is Rs. 4-14-8.

If the sum of money in Rs. be known correct to 3 decimal places, say it is given as Rs. 4·923 correct to 3 decimal places, it will be found on a similar working that the greatest and the least possible values of the sum expressed correct to a pie will be the same.

(The verification of this statement is left as an exercise to the student).

It should, therefore, be noted that any sum of money expressed as the decimal of a rupee correct to 3 decimal places is expressed correct to a pie.

Exercise VI (a).

1. The area of a field correct to a hundredth of a sq. yd. is 358 34 square yards. Between what values does the actual area lie?

2. The volume of a solid is 93·287 c.c. correct to a thousandth of a c.c. Between what values does theactual volume lie

3. A sum of money is given as £4·038 correct to three places of decimals. Write down the greatest and the least possible values of the given sum, and by working out each of these to the nearest penny show that the actual sum of money correct to the nearest penny is the same in both cases.

4. Show that in expressing a sum of money in pounds correct to three places of decimals, the possible error is less than a farthing.

5. A length is 6·8039 miles correct to four places of decimals. Find the difference between the greatest and the least possible values of the length.

CONTRACTED ADDITION AND SUBTRACTION.

§ **39.** We will indicate here how far the processes of addition and subtraction may be simplified by the omission of figures of which we have no certain knowledge or which are unnecessary for our purpose, and will also determine the possible errors due to the figures so omitted.

Ex. 1. Four numbers, known correct to 3 decimal places, are 1·418, 9·357, 8·654, 6·317. Determine their sum and indicate the limits of error.

	A	B	C
The no. corresponding to	1·418 lies between	1·4175 and	1·4185
„	9·357	9·3565 „	9·3575
„	8·654	8·6535 „	8·6545
„	6·317	6·3165 „	6·3175
	Total...25·746	25·7440	25·7480

The addition is performed as usual at A the sum being 25·746 This is not necessarily the correct sum of the given numbers. The fourth place in the given lengths might be in every case 5 and the 4 fives added would produce 20 in the fourth place, *i.e.*, 2 in the third place; therefore the third place, as shown at A may be 2 in excess as at B or 2 in defect as at C.

The answer may be written thus $25{\cdot}746 \pm {\cdot}002$.

Ex. 2. The following numbers are each correct to a thousandth. Find their sum, indicate the limits of error and find to how many figures this sum can be trusted 25·376; 14·020; 117·527, and 79·009.

Adding up the numbers the sum is found to be 225·932. Since the number corresponding to 25·376 lies between 25·3755 and 25·3765, the maximum error is 0·0005. So also

with the other numbers So, the aggregate of the errors is $4 \times 0{\cdot}0005 = 0{\cdot}002$; the sum lies between $235{\cdot}932 + 0{\cdot}002$ and $235{\cdot}932 - 0{\cdot}002$; *i.e.*, between $235{\cdot}934$ and $235{\cdot}930$. Therefore, in the sum $235{\cdot}932$, the last digit 2 cannot be trusted. So the sum is correct only to 2 places of decimals.

We may note the following facts :—

If 10 numbers, each of which is correct to, say, 3 places of decimals, be added together, the last digit of the sum may differ from the corresponding digit of the exact sum by as much as $10 \times 0{\cdot}5$ or 5 ; and if 20 such numbers be added, the maximum error may be so great as to affect the digit which is the last but one ; the number of correct figures in the sum will be reduced by two.

Ordinarily, the errors never amount to so much.

Ex. 3. Two numbers known correct to 3 decimal places are $6{\cdot}875$ and $3{\cdot}983$; determine the limits of error in taking $6{\cdot}875 - 3{\cdot}983$ for the difference between the numbers.

A	B	C
$6{\cdot}875$	$6{\cdot}8745$	$6{\cdot}8755$
$3{\cdot}983$	$3{\cdot}9835$	$3{\cdot}9825$
$2{\cdot}892$	$2{\cdot}8910$	$2{\cdot}8930$

It must be noticed that the minimum value of the difference arises by subtracting the greatest possible value of the subtrahend from the least possible value of the minuend as at B ; and the maximum value arises by subtracting the least possible value of the subtrahend from the greatest possible value of the minuend. Hence the limits of error in taking $(6{\cdot}875 - 3{\cdot}983)$ or $2{\cdot}892$ for the difference are $\pm {\cdot}001$.

In certain cases our information may extend beyond the number of places required in the answer in which case some figures in the given numbers may be altogether omitted ; the following are two examples in illustration of such cases.

Ex. 4. Find the sum, correct to 3 places of decimals of $3\cdot16\dot{6}$; 4·029668; 17·85026 and 0·007864.

Since we are asked to find the sum correct to 3 places of decimals, we ought to know enough about the 4th place to be able to say whether it is greater or less than 5, and we therefore retain 5 places, the 5th merely to know what figure is to be carried to the 4th; in setting down each number, we also correct the 5th place, as shown in the margin.

3·166	67
4·029	67
17·850	26
·007	86
25·054	46

As the fourth decimal figure in the sum is 4, the previous digit 4 need not be corrected and the sum is taken as 25·054.

We should not require any figures to the right of the vertical line but for the reason that we have to ascertain the *exact* digit in the 4th place to see if the sum approximates to 25·054 or to 25·055. It will not do if we work to *one* more figure instead of to *two*. For suppose we retained only one additional figure in each of the numbers in the above question. They would stand as shown in the margin and the sum would be 25·0546, or 25·055 correct to 3 places of decimals; whereas it is really 25 054.

3 166	7
4·029	7
17·850	3
·007	9
25·054	6

The reason for this discrepancy is evident. In writing the above numbers correct to 4 places of decimals, the errors introduced were all of the same kind, each being excess. So the sum of the errors was so great as to seriously affect the accuracy of the answer. To guard against such accumulation of errors of the same kind, therefore, the student should always work questions on addition with at least two additional figures

Even two additional figures will sometimes be found to be insufficient as when (1) the additional figures add up to 49;

(2) a very large number of quantities are to be added, say, 30. In case (1), the last digit, 9, is doubtful, and so the last two figures may actually be 48, or they may be 50. In the latter case, the digit to the left of 4 in the answer will have to be corrected. In case (2) the errrors may be of the same kind in such a large number of the quantities as to affect the last two digits of the sum.

Ex. 5. Find the difference, correct to 4 places of decimals, between 19·0784593 and 13·571428.

As in addition, two additional figures are retained in each number; and each is written correct to six places of decimals. The fifth decimal figure in the difference is 3. So the answer is 5·5070 correct to 4 places of decimals.

19·0784	59
13·5714	29
5·5070	30

Ex. 6. Find, correct to 3 places of decimals, the sum of

$$\frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \text{etc.}$$

Since the sum is required only to 3 places of decimals, we shall retain 5 decimal figures in every number, the fifth place being corrected. $\frac{1}{7} = 0{\cdot}142857\ldots$; $\frac{1}{7^2}$ is got from this by dividing it by 7; $\frac{1}{7^3}$ is got by dividing this last result by 7 and so on.

We do not consider fractions lower than $\frac{1}{7^5}$, for such fractions do not affect the first four places after the decimal point. Thus the sum is 0·167.

[Since $\frac{1}{6} = 0{\cdot}166666\ldots\ldots$ it may be noted that the sum of the above series approaches $\frac{1}{6}$]

$\frac{1}{7}$	= 0·142	86
$\frac{1}{7^2}$	= 0·020	41
$\frac{1}{7^3}$	= 0 002	92
$\frac{1}{7^4}$	= 0·000	42
$\frac{1}{7^5}$	= 0·000	07
The sum	= 0·166	67

Exercise VI (b).

1. The following numbers are known to be correct to the third decimal place. To what place can their sum be relied on? What are the limits of error?

(*a*) 8·327, 4·956, 10·320, 7·675.
(*b*) 3·1414, 6·835, 4·934, 5·678.

2. The following numbers are known to be correct to 4 places. To what place can we rely on their difference? What are the limits of error?

(*a*) 3·8456, 4·2576.
(*b*) 29·3751, 26·4876.

3. If the numbers in this question are known correct only to the figures given, state the limits of error of the answers:—

(*a*) $16 + 28 + 35 - 43$.
(*b*) $231{\cdot}2 - 375{\cdot}4 + 675{\cdot}3 - 3{\cdot}8$.

Find the value of

4. $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \ldots\ldots$ correct to 6 places of decimals.

5. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots$ correct to 4 places of decimals.

6. $1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1.2.3.4} + \ldots\ldots$ correct to 4 places of decimals.

7. $\frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \ldots\ldots$ correct to 4 places.

8. $\frac{1}{4} + \frac{2}{4^2} + \frac{3}{4^3} + \frac{4}{4^4} +$ etc., ... 2 places of decimals.

§ 40. Limits of error in Multiplication and Division.

Ex. 1. If the length of a wire known correct to three decimal places be ·976 metres, find the total length of 48 such wires and determine the limits of error in taking the product ·976 × 48 m. as the total length.

Ordinarily we proceed thus :—

Each wire is ·976 metres long.

∴ the total length of 48 such wires is ·976 × 48 metres,

i.e., 46·848 metres.

Here the length of a wire is not given to be exactly ·976 m.; but it is ·976 m. correct to 3 decimal places.

∴ the actual length may be anything between ·9755 m. and ·9765 m.

∴ we can only say that the total length must lie between ·9755 × 48 m. and ·9765 × 48 m.,

i.e., between 46·824 m. and 46·872 m.

Thus the figures in the answer we have got, *viz.*, 46·848 m. are trustworthy only up to the 8 in the first decimal place and there is a possible error of ± ·024 m.

This possible error we could have found thus :—

The length of a wire is known only correct to 3 decimal places,

i.e., there is a possible error in its length to the extent of ± ·0005 and ∴ the possible error in 48 such wires is ± ·0005 × 48 = ± ·024 m.

Ex. 2. The numbers 2·723 and 3·845 are known correct to 3 decimal places only. How far can their product be relied upon?

The number which is 2·723 (correct to three decimal places) lies between 2·7225 and 2·7235.

Similarly the number denoted by 3·845 lies between 3·8445 and 3·8455.

Hence the product of the two numbers lies between 2·7225 × 3·8445 and 2·7235 × 3·8455,

i.e., between (2·723 — ·0005) (3·845 — ·0005)

and (2·723 + ·0005) (3·845 + ·0005),

i.e., between $2{\cdot}723 \times 3{\cdot}845$
$- {\cdot}0005\,(2{\cdot}723 + 3{\cdot}845) + {\cdot}00000025$,
and $2{\cdot}723 \times 3{\cdot}845$
$+ {\cdot}0005\,(2{\cdot}723 + 3{\cdot}845) + {\cdot}00000025$,
i.e., between $(2{\cdot}723 \times 3{\cdot}845 + {\cdot}00000025) - {\cdot}0005 \times 6{\cdot}568$
and $(2{\cdot}723 \times 3{\cdot}845 + {\cdot}00000025) + {\cdot}0005 \times 6{\cdot}568$.
$\therefore$ the product $= 2{\cdot}723 \times 3{\cdot}845 + {\cdot}00000025 \pm {\cdot}003284$.

Hence the product of $2{\cdot}723$ and $3{\cdot}845$ both of which are given to be correct to 3 decimal places is correct to 2 decimal places only, and the error in the third decimal place is not greater than 3. The limits of error correct to 3 decimal places are $\pm {\cdot}003$.

NOTE.—It should be noted, from the above examples, that if two numbers are known to be correct to, say, 4 and 6 significant figures*, their product is not known to any greater degree of accuracy than the smaller number 4 of figures.

Ex. 3. Divide $18{\cdot}32$ by $2{\cdot}34$ each being known correct to two decimal places. Find the possible error in the quotient found by ordinary division.

$\frac{18{\cdot}32}{2{\cdot}34} = 7{\cdot}82906$ by the ordinary process of division.

Now the number denoted by $18{\cdot}32$ lies between $18{\cdot}315$ and $18{\cdot}325$,

and the one denoted by $2{\cdot}34$ lies between $2{\cdot}335$ and $2{\cdot}345$.

$\therefore$ the greatest quotient is got from $\frac{18{\cdot}325}{2{\cdot}335}$ and the least from $\frac{18{\cdot}315}{2{\cdot}345}$.

* The first significant figure in a number is the figure on the extreme left which is not a cypher.

But $\frac{18\cdot325}{2\cdot335} = 7\cdot847\ldots$ and $\frac{18\cdot315}{2\cdot345} = 7\cdot810\ldots$

$\therefore$ the answer $= 7\cdot829 \pm \cdot019$.

the error is in the second decimal place.

From the examples worked above, it will be seen that when two numbers, which are known to be correct only to a certain number of places, are multiplied together or divided one by the other, the result can be relied on only to a limited number of places; and we proceed to explain how to shorten our operations so as to give only a specified number of places in the product or quotient. The process is known as **Contracted Multiplication** or **Contracted Division.**

§ **41. Rough approximation to a product or a quotient.** Before we proceed to the contracted methods it will be useful for us to ascertain approximately the result of a given multiplication or division.

And the principles involved in the following algebraical truths will be frequently applied in arriving at such approximate results :—

i. If x is a very small fraction, x^2, x^3, etc., are still smaller fractions, and, compared with x, they are negligible.

ii. If x and y are small fractions, their product xy is much smaller than either x or y and is therefore negligible compared with x or y.

iii. Ordinary process of division gives $\frac{1}{1-x} = 1 + x + x^2 + x^3 +$ etc. The division does not terminate; but if x is a small proper fraction each higher power of x is very small compared with the next lower power. So, if we were to write $\frac{1}{1-x}$ as being equal to 1, the error would be $x + x^2$

$+x^3+$ etc., and may be considered too big. But if we write $1+x$ for $\frac{1}{1-x}$, the error is $x^2 + x^3 +$ &c. and is not so great, and the approximation is closer than in the former case.

Similarly, $1+x+x^2$, $1+x+x^2+x^3$, etc., will be closer and closer approximations to the correct value of $\frac{1}{1-x}$.

For practical purposes, especially when x happens to be very small, say one-thousandth, the approximation $1 + x$ for $\frac{1}{1-x}$ will be quite sufficient.

iv. If 1 be divided by $1+x$, x being a small proper fraction, the quotient $= 1 - x + x^2 - x^3 +$ etc. Here also the division does not terminate; and the values 1, $1-x$, $1-x+x^2$, etc., are approximate values of $\frac{1}{1+x}$, that value which contains a larger number of terms being a closer approximation to the exact value, namely $\frac{1}{1+x}$.

v. If x and y are small proper fractions, $(1+x)(1+y)$ may be taken as $1+x+y$. The exact value is $1+x+y+xy$; and xy is negligible as compared with x and y.

vi. $(1+x)(1-x) = 1 - x^2$ and if x is a small proper fraction, x^2 will be small when compared with x, consequently, the product $(1 + x)(1 - x)$ is taken to be 1, to the same degree of approximation as was aimed at in the above examples.

From this also we infer that

$$\left.\begin{aligned} \frac{1}{1+x} &= 1 - x \\ \text{and } \frac{1}{1-x} &= 1 + x \end{aligned}\right\} \text{ approximately.}$$

vii. By repeated multiplication we find that, $(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 = 1 + 5x$ approximately, x being a small proper fraction.

We might generalize and say that, when x is small such that the second and higher powers of x are negligible, $(1 + x)^n = 1 + nx$ approximately.

Thus $(1 + x)^7 = 1 + 7x$ approximately.

Also $(1 - x)^n = 1 - nx$ approximately.

Thus $(1 - x)^5 = 1 - 5x$ approximately.

In conformity with the above principles, the product of two numbers a and b, *viz.* ab may be taken as being roughly equivalent to $a(1+x) \times b(1 - x)$, x being a small fraction such that its square and higher powers may be neglected; for $(1+x)(1-x)$ approximates to 1.

Also $\frac{a}{b}$ is exactly equal to $\frac{a(1+x)}{b(1+x)}$. It will be wrong to take it as approximately equal to $\frac{a(1+x)}{b(1-x)}$; for $\frac{a(1+x)}{b(1-x)}$ $= \frac{a}{b}(1 + x)(1 + x)$ approximately $= \frac{a}{b}(1 + x)^2 = \frac{a}{b}(1+2x)$ approximately; which is not the same as $\frac{a}{b}$, since $2x$ does not belong to the orders we have agreed to neglect.

Ex. 1. Evaluate approximately 6894×492.

$$6894 \times 492 = 7000\left(1 - \frac{106}{7000}\right) \times 500\left(1 - \frac{8}{500}\right)$$

$$= 7000 \times 500\left(1 - \frac{106}{7000} - \frac{8}{500}\right) \text{ approximately}$$

{ for $(1 - x)(1 - y) = 1 - x - y$ approximately }

$= 3500000\ (1 - 2\%\ - 2\%)$ roughly

$= 3500000\ (1 - 4\%)$ „

$= 3360000$ roughly.

In other words multiply 7000 by 500 and deduct 4 °/₀ of the product.

Ex. 2. Evaluate approximately $\frac{765{\cdot}384}{101}$

$$\frac{765{\cdot}384}{101} = 765\,384 \times \frac{1}{100(1 + \frac{1}{100})}$$

$$= \frac{765{\cdot}384}{100}\left(1 - \frac{1}{100}\right) \text{approximately}$$

$$\left[\text{for } \frac{1}{1+x} = 1 - x\right]$$

$$= 7{\cdot}65384 - {\cdot}0765 \text{ approximately}$$

$$= 7{\cdot}577.$$

In other words divide by 100 and decrease by 1 °/₀.

Ex. 3. Evaluate $\frac{6846 \times 392}{102 \times 497}$

$$\frac{6846 \times 392}{102 \times 497} = \frac{7000 \times 400}{100 \times 500} \times \frac{(1 - \frac{154}{7000})(1 - \frac{8}{400})}{(1 + \frac{2}{100})(1 - \frac{3}{500})}$$

$$= \frac{70\not{0}\not{0} \times 4\not{0}\not{0}}{1\not{0}\not{0} \times 5\not{0}\not{0}}\left(1 - \frac{154}{7000} - \frac{8}{400} - \frac{2}{100} + \frac{3}{500}\right)$$

approximately

$$= 56\,(1 - 2\,°/_0 - 2\,°/_0 - 2\,°/_0 + 1\,°/_0) \text{ roughly}$$

$$= 56\,(1 - 5\,°/_0) \quad \text{,,} \quad \text{,,}$$

$$= 56 - 2{\cdot}8 = 53 \text{ roughly.}$$

Ex. 4. Evaluate $\frac{(864)^3}{(324)^2(298)}$

$$\text{The expression} = \frac{\{800 \times (1 + \frac{64}{800})\}^3}{\{300(1 + \frac{24}{300})\}^2 \{300(1 - \frac{2}{300})\}}$$

$$= \frac{800^3}{300^2 \times 300} \times \frac{(1 + \frac{64}{800})^3}{(1 + \frac{24}{300})^2 (1 - \frac{2}{300})}$$

$$= \tfrac{512}{27} \times (1 + 3 \times \tfrac{64}{800})(1 - 2 \times \tfrac{24}{300})(1 + \tfrac{2}{300})$$

$= 19(1 + 3 \times \frac{64}{800} - 2 \times \frac{24}{300} + \frac{2}{300})$ roughly
$= 19(1 + 3 \times 8\% - 2 \times 8\% + 1\%)$
$= 19(1 + 9\%)$
$= 20{\cdot}7$ roughly.

Ex. 5. Simplify

£1 13*s.* 6*d.* ÷ 89.

The result $= \frac{33\frac{1}{2}}{89}$*s.* roughly

$= \frac{33}{88}$*s.* roughly

$= \frac{3}{8}$*s.* or $4\frac{1}{2}$*d.* roughly.

Ex. 6.—The volume V of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$ where, $\pi = 3{\cdot}1416$ and r is the radius. Find the volume of the earth in cubic feet, the radius being 3959 miles.

From the formula, $V = \frac{4}{3} \times 3{\cdot}1416 \times (3959 \times 5280)^3$ cubic feet; which $= \frac{4}{3} \times 3{\cdot}1416 \times (4000 \times 5000)^3$, very roughly

$= 4 \times 4^3 \times 5^3 \times 10^{18}$
$= 4 \times 8 \times 10^{21}$
$= 3 \times 10^{22}$ cubic feet, roughly.

Exercise VI (c).

In the following questions, find the answer very roughly, with as little working as possible.

1. Decimalize approximately—

(*a*) $\frac{364}{102}$. (*b*) $\frac{974}{204}$. (*c*) $\frac{872}{99}$. (*d*) $\frac{2088}{808}$.

(*e*) $\frac{7{\cdot}93}{10{\cdot}3}$. (*f*) $155 \times 4{\cdot}25$.

Find roughly the value in decimals of

2. $\frac{1748 \times 8576}{24356 \times 693}$. **3.** $(0{\cdot}00762)^2 \times (15{\cdot}29)^8$.

4. $(1{\cdot}00086)^2$.

5. $\dfrac{0{\cdot}648 \times 0{\cdot}725 \times 11{\cdot}8 \times 89.}{850{\cdot}}$

6. The following numbers are known to be correct to 3 places. To what place can we rely upon: (i) their product; (ii) the quotient of the greater divided by the less; (iii) the quotient of the less divided by the greater? State the possible error in each case.

(*a*) 218·387; 21·467.

(*b*) 29·865; 13·343.

7. Sound travels at the rate of 1120 ft. per second. If the report of a gun be heard 31 seconds after the flash is seen, find roughly the distance at which the gun is fired.

8. The density of a substance is measured by its mass divided by its volume. Compare the densities of the following four substances:—

Mass in grams.	Volume in cubic cm.	Mass in grams.	Volume in cubic cm.
(*a*) 2542	289	(*c*) 823	66
(*b*) 756	92	(*d*) 1043	85

9. The value of imports into British India during the 8 months ended August 1908 was £60,848,000, while the value of the same for the corresponding period of 1907 was £54,644,000. The values of exports for the same periods in 1908 and 1907 were respectively £65,597,000, and £87,302,000. Roughly estimate the increase or decrease per cent. in the values of exports and imports in 1908.

10. A rectangular field is 456 yds. long and 348 yds. broad. Find, approximately, the area of the field in acres.

11. A train performs a journey of 192 miles in 9 hours 25 min. Find the average speed in miles per hour.

12. A sovereign weighs 123·47 grains. Find roughly how many sovereigns may be coined out of 47 lbs. standard gold, each lb. weighing 5760 grains.

13. The area of the southern districts of the Madras Presidency is 53,578 square miles; while the population is 19,366,486. Find, roughly, the population per sq. mile.

14. The Excise receipts of the Madras Presidency for 1904-05 are divided between Imperial and Provincial Funds as Rs. 14.065,365 and Rs. 4,688,455. What percentage of the receipts goes to the Provincial Funds?

15. If 3,466 marakkals = 10,000 gallons nearly, how many gallons are roughly equal to 8,455 marakkals?

§ 42. Contracted Multiplication.

Ex. 1. Multiply 8·52467 by 6, correct to two places of decimals.

Explanation. We require *two* decimal places; therefore we retain 3 places in the multiplicand (so that we may know whether the 3rd place in the product is greater or less than 5 with a view to correct the 2nd place) and draw a line to the right of them to show that the rest are not wanted. We begin the work of multiplication upon 4 neglecting the figures on the right except to show us what to carry from right to left. It is clearly 3; and the rest of the work proceeds as usual. The answer is 51·15 (correct to 2 places).

```
8·524 | 67
6     |
------|---
51·147| ß
```

Ex. 2. Find, correct to 2 places of decimals the product of 46·05383 and 29·37423.

46·05383 × 29·37423 = 460·5383 × 2·937423 (reducing the multiplier to the standard form, *vide* p. 213, Part I) = 1380, roughly, *i.e.*, the integral portion consists of 4 significant figures. This rough approximation determines the position of the decimal point in the product. Since the answer

must be correct to 2 places of decimals, there will be 6 significant figures in the answer.

460·5383	2·937423	4605383	2937423
921·0766		9210766	
414 4844	7	4144845	
13 8161	49	138161	
3·2237	681	32237	
·1842	1532	1842	
·0092	10766	92	
13	816149	14	
1352·7957	948009	1352796	

The answer = 1352·80 correct to two places.

The working is shown completely on the left. But since we want the product only to 6 significant figures, the figures to the right of the vertical line are useless. Even the column of figures immediately to the left of the vertical line is wanted only to know definitely the seventh significant figure in the product with a view to correct the 6th figure if necessary.

The working on the right shows how the work can be contracted without any loss of accuracy; and this contracted work we proceed to explain:

(1) The position of the decimal point in the product should be checked by some independent method as we have done above. In the working we have, therefore, discarded the decimal point altogether.

(2) We work the product to seven figures, *i.e.*, to one place more than the required number of significant figures, hence we retain 7 figures in the multiplicand, the last figure being corrected.

It does not matter how many figures we put in the multiplier, as will be seen from the explanation of the process given below.

(3) The actual process of contracted multiplication may be explained thus :

The multiplicand is multiplied by 2, the operation beginning with the highest figure of the multiplier, *i.e.*, from the left and proceeding to the right.

We then multiply by the 9. While multiplying by the 9, we reject from the multiplicand the last figure remaining on the right, *i.e.*, 3, using it however to supply the carrying figure to the first figure set down.

Next we multiply by the 3 but while multiplying by the 3 we reject from the multiplicand the last figure remaining on the right, (*i.e.*, 8), using it however to supply the carrying figure to the first figure set down. In this case 3 times 8, *i.e.* 24, which is nearer 20 than 30, we carry 2 to the first figure.

We next multiply by the 7 and while doing so, we reject from the multiplicand the last remaining figure on the right, *viz.*, 3, using it however to supply the carrying figure.

Continuing the same principle we find that the first figure set down in each line of work has the same local value.* The partial products then are added.

---

* Let us suppose that 3 in the multiplicand is in a place which has the local value $(10)^m$, and 2 in the multiplier has the local value $(10)^n$.

Then the 6 arising in the product of 2 and 3 must occupy a place having the local value $10^m . 10^n$ or 10^{m+n}.

Now 8 and 9 are multiplied, their local values are 10^{m+1} and 10^{n-1}.

$\therefore$ the local value of the 2 arising in the product of 8 and 9 $= 10^{m+1} . 10^{n-1} = 10^{m+1+n-1} = 10^{m+n}$.

Hence to get figures of the same local value to be written in the first vertical line on the right, figures equally distant from 3 and 2 (the starting figures) in the multiplicand and the multiplier should be selected for multiplication—which accounts for the multiplication work becoming lighter and lighter as we proceed to the end.

[As each digit is used in the multiplier, the student is advised to put a dot over it to record that it has been used; and also to score out the corresponding digit of the multiplicand to indicate that it is no longer wanted.]

Ex. 3. Find, correct to 4 places of decimals the product of 123·507 and ·0007835.

Now $123{\cdot}507 \times {\cdot}0007835 = {\cdot}0123507 \times 7{\cdot}835$
$= {\cdot}01 \times 8$ (roughly) $= {\cdot}08$.

It is clear that there is a zero between the decimal point and the first significant figure. Since 4 places are required in the product, we require 3 significant figures and let us work out to 4 figures, discarding decimals in the two numbers.

The product is ·0968.

```
1235
   7835
-----
8645
 988
  37
   6
-----
 968
```

NOTE.—In the examples worked above, we have retained in the multiplicand one more figure than the number of figures required in the product, the last figure being corrected. But it would be easy to construct a product, as we have seen in the case of addition, for which this method of approximation appeared incorrect by a unit in the last figure. The danger of this error would be lessened, though it cannot be entirely removed, if two more places were retained in the multiplicand than are required in the product. But the method shown above is more than sufficiently exact in practice, though it may not meet *critical* cases which one may choose to manufacture.

Ex. 4. Multiply 9734869·58 by 68·3475 correct to a million.

It is enough in this case if the multiplicand is expressed as millions, *i.e.*, as a decimal of 10^6 (for 10^6 is a million).

$$9734869{\cdot}58 = 9{\cdot}73486958 \times 10^6.$$

$$\therefore\ 9734869{\cdot}58 \times 68{\cdot}3475$$

$$= 9{\cdot}73486958 \times 68{\cdot}3475 \times 10^6$$

$$= 10 \times 68 \text{ or } 680 \text{ millions roughly.}$$

It is enough if the first three significant figures are obtained.

$$\begin{array}{r} 6835 \\ 973486958 \\ \hline 61515 \\ 4785 \\ 205 \\ 27 \\ 5 \\ \hline 66537. \end{array}$$

The product is $665 \times 10^6 = 665{,}000{,}000$.

Exercise VI (d).

Evaluate the following correct to two decimal figures :—

1. $6{\cdot}3285 \times 3{\cdot}4294$. **2.** $83{\cdot}8932 \times 17{\cdot}6745$.

3. $\cdot03898 \times 6{\cdot}4387$.

Find correct to tbe nearest unit the product of —

4. 46·893 and 79·3854. **5.** 108·3874 and 498·365.

Find the value of the following correct to three significant figures :—

6. $8979 \times 6075{\cdot}89$. **7.** $6875{\cdot}38 \times 4870{\cdot}5$.

8. $\cdot0007485 \times \cdot06357$. **9.** $\cdot00004675 \times 983{\cdot}75$.

Find the following products :—

10. 86756×40895 to the nearest million.

11. $\cdot000875$ by $8{\cdot}35 \times (10)^7$ to the nearest $(10)^{10}$.

12. Given that 1 metre = 39·3701 inches, find correct to the nearest unit the number of inches in 8·64 metres.

13. Taking a centimetre as $\frac{2}{5}$ of an inch, find to the nearest unit the number of centimetres in 4 yds. $6\frac{3}{5}$ inches.

14. A train runs at an average rate of 45·3 miles per hour. Express this rate in km. per hour to the nearest unit., given 1 mile = 1·609 km.

15. Find to the nearest pie the value of a bar of gold $3\frac{1}{2}$ metres long at Rs. 15-8-0 per inch (1 metre = 37·37079 inches).

§ **43. Contracted Division.** In connection with contracted division as in contracted multiplication there are two questions involved, *viz.*, (1) how far the quotient could be relied on ; (2) how far we want the quotient. And we can contract our work according to our data and to our requirements and the process will become clear by considering a few examples.

Ex. 1. Divide 0·754327892 by 0·047232128, ; (1) to as many places as the quotient could be relied on (2) correct to three places of decimals.

The quotient is the same as in $\frac{75{\cdot}4327892}{4{\cdot}7232128}$ and so its integral portion will consist of 2 figures. The full working according to the ordinary process of division is shown below on the left and the contracted work is shown on the right.

15·97065	2	*Ans.*
47232128)7543278	92	
4723212	8	
2820066	12	
2361606	40	
458459	720	
425089	152	
33370	5680	
33062	4896	
308	078400	
283	392768	
24	6856320	
23	6160640	
1	06956800	
	94464256	

1597065	
47232128)75432789	
47232128	
28200661	
23616064	
4584597	
4250891	
333706	
330625	
3081	
2834	
247	
236	
11	

If the given numbers are known correct only to the figures given, all figures which occur in the ordinary process of division to the right of the vertical line drawn are doubtful and hence the process involved in finding those figures is mere waste. Hence the usefulness of a contracted method which will do away with all these doubtful figures. Such a method is shown on the right. Since the answer can be relied on only up to the figure 5 which is immediately to the left of the vertical line, in the contracted method we work to one more figure. The method may be briefly explained thus :—

In the process, we do not, at every successive step, bring down any figures from the dividend as is the case in ordinary division, nor do we add zeroes when the figures of the dividend have all been taken up. On the other hand we cut off a digit from the right of the divisor after each subtraction Thus after the first subtraction we cut off 8 from the divisor ; but while forming the partial product corresponding to 5, we carry 4 ; because 5×8 (the figure cut off) is 40 and 4 is to be carried. So on for the other figures successively cut off, the product determining the figure to be carried being first duly found. The remainder after the last 5 of the quotient was got is 11 and the next figure of the quotient would be 2 ; and hence 5 is not to be corrected.

(2) Since the quotient must be expressed correct to 3 places of decimals, we should obtain 5 significant figures in the quotient. For this purpose, it is enough if we retain $5 + 1$ or 6 significant figures in the divisor, and duly correct the sixth figure. We retain as many figures in the dividend as are required for the 1st step in the process of division by the altered divisor. Here also we should correct

the last digit of the dividend that is retained. If there are not enough figures in the dividend for this division, we add zeroes. Thus the division reduces itself to the division of 754328 by 472321, the decimal points being discarded. The working is shown in the margin below.

```
      15971
47232 ] 754328
        472321
        282007
        236161
         45846
         42509
          3337
          3306
            31
```

After the 1st significant digit of the quotient has been determined, there are only *four* more digits to be found, and so, it is enough if we retain *five*, instead of all the six figures of the divisor, for subsequent division. Thus the divisor at the second stage of the division here is 47232. However to the partial product of 47232 and the 5 of the quotient, we carry 1, for 5×1 (the figure that was cut off) is 5, which is corrected into 10, and the 1 is carried. So, also, to the 3rd partial product, 2 is carried; for 9×2 (the figure that was cut off) $= 18$ which is nearer 20, and 2 is carried.

The last remainder is 31 and the divisor is 47; the corresponding digit of the quotient is zero; but since 31 is nearer to 47 than to zero, the last digit of the quotient is corrected into 1.

Since we have determined that the integral portion of the quotient consists of 2 figures, the required answer is 15·971.

Ex. 2. Divide 25·64 by ·3654 to one place of decimals.

Here the number of digits in the integral portion of the quotient is obviously two, and as the result is required to one place of decimals, the first three figures of the quotient ought to be determined.

We have therefore to retain in the divisor one more digit than three,—that is four figures.

```
    7017
3ϐϐ4) 25640
      25578
      ----
         62
         37
         --
         25
```

Thus the answer required is 70·2.

Ex. 3. Divide 48·062543 by 7006·32 to 9 places of decimals

$$\frac{48\cdot062543}{7006\cdot32} = \frac{\cdot048062543}{7\cdot00632} = \cdot007 \text{ (roughly).}$$

In the quotient two ciphers will follow the decimal point. Hence *seven* figures must be got by division.

According to the rule, 8 figures should be taken in the divisor and in the dividend as many as are needed for the first step of division. There are only 6 places in the divisor. So two ciphers should be added and in the dividend a cipher should be added to begin the first step of division. ...The rest of the working is as shown in example (1). ∴ the required quotient is ·006859884.

```
      685988407
7ϐϐϐ4ϐϐ)480625430
        420379200
        ---------
         60246230
         56050560
         --------
          4195670
          3503160
          -------
           692510
           630569
           ------
            61941
            56051
            -----
             5890
             5605
             ----
              285
              280
              ---
                5
```

Instead of adding ciphers both to the divisor and to the dividend the contracted division may be better performed in the following manner: Suppose the divisor were written with 8 figures and division by contraction were done as shown above. Then, our divisor with the actual six figures

would appear as an operator only in determining the 3rd figure of the quotient; *i.e.*, we begin cancelling figures of the original divisor only after we have found the 3rd figure of the quotient. Hence the division might be done by the ordinary method for the 1st 3 figures of the quotient and then contraction might be employed for the other figures. And generally where the divisor contains q digits and the required number of figures in the quotient is p, $(p > q)$, $p - (q - 1)$ figures are to be obtained by ordinary division and $(q - 1)$ figures by contraction.

```
      68598841
70ϕβχ)48062543
      4203792
      6024623
      5605056
       4195670
       3503160
        692510
        630569
         61941
         56051
          5890
          5605
           285
           280
             5
```

Exercise VI (e).

Obtain the following quotients true to four significant figures in each case:—

1. 3·14157 ÷ 2·3857. **2.** 2·142857 ÷ 14·2134.

3. 864·356 ÷ 43·064. **4.** 36·4285 ÷ 418·354.

Find the following quotients correct to the third decimal figure:

5. 845·00038 ÷ 34·4567. **6.** 169·0384 ÷ 6057·28.

7. ·078467 ÷ 36·03. **8.** 1·3257 ÷ 160·498.

Evaluate the following to the fourth significant figure:—

9. 1 ÷ 18·39857. **10.** 1 ÷ 3·14159.

11. 3 ÷ 1·83507. **12.** $8\dot{2}\dot{8} \div 1\dot{9}2\dot{8}$.

Decimalize the following fractions using contracted division and giving the results true to 4 significant digits:—

13. $\frac{127}{1395}$. **14.** $\frac{691}{2085}$.

From the following equations determine the value of x correct in each case to the nearest tenth.

15. ·06875596 × x = 40·385714.

16. 8935x = 183478.

17. If 1 mile = 1·609 kilogrammes, express a metre in inches to four significant figures.

18. Given that one metre = 39·3701 inches, express 1 inch in metres and hence in millimetres, correct to 4 significant figures.

19. If a lunation be 29·5306 days and a tropical year 365·242216, find the number of lunations in a tropical year to the first three significant figures.

20. The circumference of a circle is 3·14159 times the diameter. The mean circumference of the earth is 43795152 yds. Find the mean diameter of the earth in yards to 5 significant figures.

EXAMPLES ON CONTRACTED MULTIPLICATION AND DIVISION (*combined.*)

Ex. 1. Find the value of

83·6789 × 3·8343 × 10·0393 to five significant figures.

(Rough estimate:

The product = 836·789 × 3·8343 × 1·00393
= 800 × 4 × 1 = about 3200). Thus the integral portion consists of 4 figures.

```
 836789
     38343
 -------
 2510367
  669431
   25104
    3347
     251
 -------
 320850.
     100393
 -------
 320850
    963
    289
     10
 -------
 322112
 -------
```

The required answer is 3221·1.)

N.B.—The student may verify the result by working it out without reducing the multipliers to the standard form.

Ex. 2. Evaluate

$$\frac{87{\cdot}384 \times 0{\cdot}6357}{3{\cdot}435}$$ to the first decimal place.

(Rough estimate

The given expression

$$= \frac{8{\cdot}7384 \times 6{\cdot}357}{3{\cdot}435}$$

$$= \frac{9 \times 6}{3}$$ about 18).

$\therefore$ the answer has two figures in the integral part.

We must get the result correct to three significant figures.

$\therefore$ we work the numerator correct to 5 figures.

The working is shown below,

87384	1617
6357	3435)5555
524304	2120
26215	59
4369	25
611	1
55550.	

The answer is 16·2.

Exercise VI (f).

1. Find the value of

(*a*) $(1{\cdot}3087)^2$ correct to 3 decimal places.

(*b*) $(39{\cdot}3701)^3$ correct to 4 decimal places.

2. A Railway Company working in the Presidency of Madras has $1261\frac{3}{4}$ miles of railway. Its takings for a week are Rs. 10,853,467. What is the average per mile to the nearest rupee ?

3. What whole number is most nearly equal to $78{\cdot}3456 \times 3{\cdot}14159$?

4. Express $\frac{1}{3{\cdot}14156}$ as a decimal fraction true to the nearest thousandth. Use the result to find to the nearest inch the diameter of a circle whose circumference is 100·98 inches.

5. Express $\frac{1}{108{\cdot}57}$ as a decimal to the nearest millionth. Use this value to find the diameter of the earth from that of the sun

which is 1380000 km. assuming that the diameters of the earth and the sun are in the ratio of 1 : 108·57.

6. The mean distance of Jupiter from the sun in terms of the earth's distance is 5·2028. The time taken by light to travel from the sun to the earth is 499 seconds and the velocity of light is 186,330 miles per second. Find the distance in miles of Jupiter from the sun to the nearest million.

7. The area of the United Kingdom of Great Britain and Ireland is 123,500 square miles and the population is about 41·2 millions. Find the average population per square mile to the nearest unit.

8. If India is sixteen times the size of Great Britain and its population about 335 millions, find the average population per square mile. Given the area of Great Britain to be 75,500 sq. miles.

9. The mean radius of the earth is $6{\cdot}37 \times 10^8$ centimetres and its mean density is 5·6. Express its mass in grammes assuming the earth to be a sphere and the volume of a sphere $= \frac{4}{3}\pi r^3$, where $\pi = 3{\cdot}14159$. Answer to three significant figures.

[*N.B.* The mass of a substance is obtained by multiplying its density by its volume.]

10. A ton of chalk occupies a volume of $15\frac{1}{2}$ cub. ft. Assuming that a mass of a cub. ft. of water is $62\frac{1}{2}$ pounds, find the specific gravity of chalk referred to water as the standard substance to the first decimal place.

11. Assuming that the Government derives a revenue of 75 crores of rupees and that the population is 356 millions, find the tax paid per head of the population.

12. Express an atmospheric pressure of 15 pounds per square inch as so many grammes per sq. centimetre. Given 1 metre = 39·3701 inches and that 1 pound = 453·6 grammes.

*__13.__ The volume of the receiver of an air-pump is 315·325 c.c. and that of the barrel 250·635 c.c. The density of the air in the receiver after n strokes is to that before exhaustion.

$$\text{as } \left(\frac{R}{R+B}\right)^n : 1.$$

Where R and B represent respectively the volumes of the receiver

and the barrel. Find to 2 places of decimals the ratio of the density after 2 strokes to the initial density.

***14.** The receiver of an air-pump has a capacity of 1 litre and the pressure of the air within it is 76·5 cm. If the barrel has a capacity of 500 c.c., what will be the pressure after 4 strokes to the nearest unit in cm.?

***15.** A piece of brass wire is exactly 3 metres long at 250°. Find its length at 0° to the nearest tenth of a cm. given

that $l_{0^\circ} = l_{250^\circ} / (1 + 250\alpha)$

where $\alpha = .000019$. l_{0° represents the length of the wire at 0° and l_{250°, its length at 250°.

***16.** The total numbers of first class, second class and third class passengers carried on a certain line of railway in a certain year were 1286,4836 and 8,354,685 respectively. The average number of miles travelled by a first, second, and third class passenger was $5\frac{1}{4}$, $3\frac{3}{4}$ and $2\frac{1}{2}$ miles, respectively. The total collections from passenger traffic were Rs. 1,45,68,750. Assuming that the first, second and third class rates per mile were in the ratio of 8 : 4 : 1, find to the nearest tenth of a pie the rate charged per mile for a third class passenger.

17. A merchant imports ironware worth Rs. 8,12,854. The imports consist of 52 consignments, each consignment divided into 1281 lots, each lot weighing 12 cwts. 2 qrs. Find to the tenth of a pie the selling price of each pound of the ironware, if he sells the articles at a profit of 25·6 per cent.

18. Find the value of the following:—

(1) $8{\cdot}3226 \times 10{\cdot}9870 \times 12{\cdot}809$ to two significant figures.

(2) $3{\cdot}1457 \times (6{\cdot}9875)^2$ to three significant figures.

(3) $\dfrac{3{\cdot}8976}{89{\cdot}352 \times 102{\cdot}365}$ to four decimal places.

(4) $\left(\dfrac{{\cdot}0849}{{\cdot}0038}\right)$ to one decimal place.

(5) $\dfrac{186867 \times 354828}{2486 \times 4308}$ to the nearest thousand.

(6) $\dfrac{4387 \times 50\frac{1}{2} \times 61\frac{1}{4}}{1780 \times {\cdot}135}$ to the nearest integer.

19. The amounts spent by two Zemindars during the periods 1909—1911 on their lands for purposes of improving the agricultural methods are given below. Find (i) the cost per acre of the improvement effected (to the nearest anna) ; (ii) the cost per acre if all the lands belonged to the same Zemindar.

1st Zemindar 24675 acres ... ... Rs. 1,68,713
2nd „ 21893 acres ... ... Rs. 1,25,315.

20. The following figures give the latest estimates of the annual collections of the special Railway cess and the actual amount to the credit of each Board on the 1st April 1911 :—

Name of District Board.	Estimated income from Railway cess in		Total amount to credit on 1st April 1911.
	1911-1912.	1912-1913.	
	Rs.	Rs.	Rs.
S. Arcot ...	69,370	68,200	4,46,691
Coimbatore ...	48,990	49,000	5,30,050
Guntur ...	1,04,510	1,04,510	8,70,929
Kistna ...	1,31,900	1,39,900	7,83,758
Kurnool ...	38,550	38,250	3,08,971
Madura ...	49,370	51,540	2,31,839
Ramnad ...	62,620	61,390	1,79,992
Salem ...	42,400	42,400	4,49,992
Tinnevelly ...	57,010	57,410	5,28,383

Find (1) how much per cent. is the increase or decrease in the income in the cess of each district in the second of these two years as compared with the income in the first year, (to the nearest integer) ; (2) the excess or defect in the cess of the second year to the nearest pie per rupee of the cess of the first year; (3) also the average total amount to credit on the 1st April 1911 to the nearest ten thousand.

21. The Government spends Rs. 8,75,680 on irrigation to afford improved agricultural facilities to the ryots of a district the arable land in which amounts to 1,815,320 acres. Find to the nearest pie the amount spent on each acre under irrigation. If

each acre is assessed at the rate of Rs. 8 and, if the Government has to pay interest at the rate of $2\frac{1}{2}\%$ on the money invested in irrigation, find to the nearest quarter anna the increase per rupee in the assessment of the district that will have to be charged, so that the Government may neither lose nor gain.

22. A bankrupt's debts are Rs. 3051-0-8. After all expenses have been paid his assets are Rs. 950, 10 a. 0 p. How much ought a man to get who is owed Rs. 48-6-8 ?

23. A kilometre being 1093·6384 yds., find to four places of decimals how many kilometres there are in 100 English miles.

24. A metre is 39·3701 in. If this is a ten-millionth part of the distance of the equator from the pole, what is the circumference of the earth the measurement being taken along a meridian ? Give the result to the nearest mile.

25. Find a decimal fraction correct to the first three places, such that if the price of a material in francs per metre be multiplied by it, the product will be the price of the same material in annas per yd. having given that Rs. 15 = 25 fr. and 1 m. = 39·37 in.

26. If a French *are* is a square with a base of 10 m. and a metre is taken as equal to 39·37 English inches, how many French ares are in an English acre to the nearest integer ?

27. A metre is 39·3708 in. Find to three places of decimals (1) the number of acres in a hectare which is 10000 sq. m. (2) the number of hectares in an acre.

28. If a cubic foot of water weighs 1000 oz., express a kilogram in pounds correct to one place of decimals (1 kilometre = 39371 inches).

29. According to the census of 1891 the population of Madras was 452,518 and that of Bellary was 900,126. Assuming that during the next decade the former increased at the rate of 5·4 per cent. and the latter decreased at the rate 20·6 per cent., express the change in percentage to one place of decimals in the population of the two districts taken together.

30. Find to two decimal places the value of

$$7{\cdot}856 - \frac{1{\cdot}489 \times 10^4}{x} - \frac{1{\cdot}456 \times 10^6}{x^2}, \text{ when } x = 273.$$

31. Find the value of x correct to the nearest unit :

(1) $\frac{x}{100{\cdot}4867} = 6{\cdot}3874 \times 4{\cdot}3875.$

(2) $x \times 108{\cdot}432 = 6008 \times {\cdot}3475.$

32. A sovereign weighs 123·27 grains and an American gold dollar weighs 33·2199 grains. Find the value in British money correct to the nearest penny of the American dollar.

§ **44. Relative and Absolute error.** If a length is actually 23·829 m. and we write it as 23·8 m. the difference between the real and the adopted value, *viz.*, ·029 m. is called the **absolute error.**

The ratio of this to the real value, *viz.*, $\frac{{\cdot}029}{23{\cdot}829}$ or $\frac{{\cdot}0029}{2{\cdot}3829}$ or approximately ·001 is called the **relative error.**

When the word ' error ' alone is used, absolute error will always be meant. Sometimes the relative error is given in the form of a percentage and is then called the **percentage error.** Thus in the last example, the percentage error is $\frac{{\cdot}029 \times 100}{23{\cdot}829} = {\cdot}001 \times 100$ or ·1 approximately.

It is very important to note what the maximum absolute error corresponding to a given degree of accuracy will be All lengths are not given correct to the nearest inch. The barometric readings are given correct to a hundredth of an inch, *e.g.*, 29·34 inches, *i.e.*, the actual value lies between 29·335 and 29·345. The maximum absolute error is ·005 either in excess or in defect and is indicated ± ·005. The length of a pole is often given to the nearest inch. The maximum absolute error is ± ·5 of an inch. The length of a house will usually be given to the nearest foot. The maximum absolute error is ± $\frac{1}{2}$ a foot. The distance between Madras and Calcutta will usually be given to the nearest mile. The maximum absolute error is ± $\frac{1}{2}$ a mile.

The diameter of the earth is often given to the nearest 100 miles. The maximum absolute error in this case is ± 50 miles.

The distance from the earth to the sun is given to the nearest million and the maximum absolute error is ± 500,000 miles. Thus the error increases with the quantity we are measuring. In the above examples the maximum absolute error varies from 0·005 in. to 500,000 miles. The relative error varies much less and is more important than the absolute error.

There is one important exception to this. In all monetary transactions results are worked to the nearest pie or penny according as the money is Indian or English, whether we are dealing with small or large sums. The maximum absolute error in all cases is $\pm \frac{1}{2}\,p.$ or $\pm \frac{1}{2}\,d$ but the relative error may be anything.

When several quantities whose values are known only approximately are denoted by the same significant figures, the maximum absolute error in each case may be different but the same maximum relative error is made. Consider the lengths 3·14 m. 314 m. and 31400 m. which are each given correct to three significant figures. The number 3·14 lies between 3·135 and 3·145; and so the maximum absolute error is 0·005. The relative error is $\frac{0{\cdot}005}{3{\cdot}14}$ $= \frac{{\cdot}5}{314}$. Again the number 314 which is correct to the last digit lies between 313·5 and 314·5; so the maximum absolute error is 0·5. But the maximum relative error is $\frac{0{\cdot}5}{314}$ which is the same as for 3·14.

Also 314 hundreds correct to the last significant digit lies between 31350 and 31450, *i.e.*, the absolute error is 50; but the relative error is $\frac{50}{31400}$; which is again the same fraction as in the previous two cases.

If in the above quantities instead of 314 we put any whole number containing three significant figures such a number will lie between 100 and 1000; the maximum relative error will be between $\frac{0{\cdot}5}{100}$ and $\frac{0{\cdot}5}{1000}$, *i.e.*, between 0·005 and 0·0005, *i.e.*, is greater than ·0005 but less than ·005.

Similarly it may be shown that in case of quantities expressed correct to four significant figures the maximum relative error is between ·0005 and ·00005 and so on. If a result is known correct to a given number of significant figures the accuracy of the value measured by the relative error is roughly known. If however the result is known correct to a given number of decimal places the accuracy of the value measured by the maximum absolute error is known. This statement can be easily verified by the student.

Exercise VI (g).

1. The weight of a body is given 384 Cg. and the error may be as much as a milligram. What is the maximum relative error? Find the percentage error.

2. Taking the equatorial radius of the earth to be 4000 miles and the maximum error to be about 41 miles, show that the relative error is about ·001.

3. A watch gains 15 seconds a day. Show that the relative error is about ·00017.

4. The ratio of the circumference of a circle to its diameter is 3·14159.....(1) If it is taken as 3·14, find the relative error.

(2) If it is taken as $3\frac{1}{7}$, what is the relative error ?

5. A metre is 39·3701 inches. If it is taken to be 39 inches, to how many significant figures is this correct ? If it is taken as $39\frac{2}{5}$ inches, to how many significant figures is this correct ? What is the percentage of error in both cases ?

6. A kilogram = 15432·356.....grains. What is the relative error if a kilogram is taken (1) to be equivalent to 15432·010 grains; (2) 15432 grains ?

7. A body weighed on three different occasions 68·77824 gm. 68·77814 gm, and 68·77856 gm. To how many significant figures do these results agree ?

8. The lengths 9300 m. 93 m. and 0·093 m. are each correct to two significant figures. Show that the maximum error is in each case the same fraction of the corresponding length.

9. If a number is correct to two significant figures, show that the maximum error is between 0·05 and 0·005 of the given number.

10. A rod whose length is 24 m. 40 cm. is measured with a metre rule (divided into centimetres) and appears to be 21 m. 45 cm. long. Supposing that the divisions of the rule are all equal, find to the nearest millimetre the error in each.

11. To convert metres into feet I multiply first by 10 and then divide by 3. I find that the error thus arising in a certain calculation is 63 ft. What is the real length assuming 1 metre = 39·37 in.?

12. One metre = 39·37011 in. Express this with 3 significant digits ; and find the percentage error in so doing.

13. Show that when a fraction of a Rupee is expressed correct to 3 decimal places, the error is less than $\frac{1}{10}$ of a pie.

14. Find the percentage error in adopting the following rule :— "To convert a rate given in miles per hour into a rate in feet per second, multiply the number expressing miles by $1\frac{1}{2}$."

15. The following table gives the value of g, or the acceleration due to the force of gravity, at different places on the earth :—

Locality.	Value of g.
Equator	32·091
Madras	32·117
Latitude 45°	32·173
Paris	32·183
Greenwich	32·191
Edinburgh	32·203
Pole	32·255

At what places does the value of g approximate to 32·2 ? Find the percentage error if g is taken as 32 at each of the above places ; and show that it is less than $\frac{4}{5}$.

16. The following values are adopted for π : $3\frac{1}{7}$, $\frac{355}{113}$, 3·1416, and 3·141592. To how many significant figures may these values be trusted ? (π = 3·14159265.....)

If the area of a circle whose radius is r, is πr^2, find the difference made by adopting the several values of π given above, to find the the area of a circle whose radius is 100 inches.

17. The weight of a heavy substance, correct to 3 significant figures, is 5270 tolas. Find the greatest and the least probable weight. Show that the maximum relative error is less than $\frac{1}{1000}$.

18. Why would you not have a viss of coffee weighed out in a balance used for weighing heavy weights such as firewood ?

19. What degree of accuracy is it most reasonable to expect in the following cases ?

(*a*) The height of a tree.

(*b*) The area of the Madras Presidency.

(*c*) The salt revenue of India.

(*d*) The thickness of the hair-spring of a watch.

(*e*) The temperature of the human body.

(*f*) The time over a race-course.

20. The land-revenue in a certain year, of the Madras Presidency is given as Rs. 5,17,79,395 against Rs. 5,51,15,285 in the previous year; or about 33·36 lakhs less than in the previous year. To how many significant figures is this last figure correct? Express the two former figures with the same degree of approximation as the last.

21. The average area of a Taluq of the Tanjore District which contains 9 Taluqs is 412 sq. miles, to the nearest sq. mile. Calculate the area of the whole District and the greatest possible error in your result.

22. The incidence of stamp duty on every head of population of the Madras Presidency is 3 as. 11 p. If the population be 36,733,533 show that the revenue from stamps is very roughly Rs. 90 lakhs.

23. Show that if a weight is given in maunds, correct to 2 places of decimals, the error is less than 2 palams.

24. In measuring angles with a protractor, a boy commits an error in each case amounting to a quarter of a degree. Show that his measurements of larger angles will be closer approximations than measurements of smaller ones.

25. A man notes the time for a single oscillation of a pendulum, as well as the time for 10 oscillations. In each case, the time is correct to the fifth of a second. Show that the relative error in the latter case is much smaller than in the former; and find it.

§ **45. Limits of accuracy.** The maximum relative error in the product or quotient of any number of quantities is approximately the sum of the maximum relative errors of the quantities when the errors are small.

Let x, y, z be three quantities and let a, b and c be the maximum relative errors respectively, these being small quantities.

Then the maximum value of $\frac{x.y}{z}$ is got by giving maximum values to x and y and minimum value to z.

$\therefore$ the values of x, y and z to be taken are $x + ax$, $y + by$ and $z - cz$.

$\therefore$ the error is $\dfrac{(x+ax)\,(y+by)}{z-cz} - \dfrac{xy}{z}$

$$= \frac{xy}{z}\,\frac{(1+a)\,(1+b)}{1-c} - \frac{xy}{z}$$

$$= \frac{xy}{z}\,(1+a)\,(1+b)\,(1+c) - \frac{xy}{z} \text{ approximately}$$

$$= \frac{xy}{z}\,(1+a+b+c) - \frac{xy}{z} \text{ approximately}$$

$$= \frac{xy}{z}\,(a+b+c).$$

$\therefore$ the relative error is $a+b+c$.

Similarly it may be shown that if we find the minimum value of $\frac{xy}{z}$ (by giving to x and y their minimum values and to z its maximum value) the relative error is $a+b+c$.

Ex. 1. Two adjacent sides of a rectangle are measured and are found to contain 1·534 m. and 0·662 m. respectively; find its area. If in each measurement there may be an error of 2 mm., what is the greatest possible error in the area of the rectangle as calculated from the above measurements?

Sol.—A millimetre $= \frac{1}{1000}$ of a metre,

$\therefore$ the maximum relative errors are $\frac{2}{1534}$ and $\frac{2}{662}$.

$\therefore$ the maximum relative error of the area, *i.e.*, in the product $1{\cdot}534 \times {\cdot}662$.

$= \frac{2}{1534} + \frac{2}{662}$

$=$ ·0013 + ·0030 roughly

$=$ ·0043 roughly.

$\therefore$ the maximum absolute error in the area

$=$ ·0043 × 1·534 × ·662 sq. m.

$=$ ·004 × 1·5 × ·7 roughly

$=$ ·004 sq. m. roughly.

In a simple example such as this, the error may be worked out more easily as follows :—

The maximum area

$= (1{\cdot}534+{\cdot}002)\,({\cdot}662+{\cdot}002)$ sq. m.

$\therefore$ the maximum error

$= (1{\cdot}534+{\cdot}002)\,({\cdot}662+{\cdot}002)-1{\cdot}534\times{\cdot}662$

$= {\cdot}002(1{\cdot}534+{\cdot}662)$ neglecting $({\cdot}002)^2$

$= {\cdot}002\times 2{\cdot}196$

$= {\cdot}004$ roughly.

The answer will contain 4 significant figures with a possible error in the fourth (the integral part containing only one figure).

$$\begin{array}{r} 1534 \\ 662 \\ \hline 9204 \\ 920 \\ 31 \\ \hline 1015 \end{array}$$

The area $= (1{\cdot}015\pm{\cdot}004)$ sq. m. roughly.

Ex. 2. If a substance weighs a grams in air and b grams in water, its specific gravity is $\dfrac{a}{a-b}$. If $a = 24{\cdot}216$, $b = 13{\cdot}195$ each correct 0·001 and there are no other errors of experiment, find the specific gravity of the substance as accurately as the figures justify.

The s. g. of the substance

$$= \frac{a}{a-b} = \frac{24{\cdot}216}{24{\cdot}216-13{\cdot}195}$$

$$= \frac{24{\cdot}216}{11{\cdot}021} = 2 \text{ roughly.}$$

The maximum relative errors are $\dfrac{{\cdot}0005}{24{\cdot}216}$ and $\dfrac{{\cdot}0005}{11{\cdot}021}$.

$$\begin{array}{r} 2{\cdot}1973 \\ 11)24216(\\ 22042 \\ \hline 2174 \\ 1102 \\ \hline 1072 \\ 992 \\ \hline 80 \\ 77 \\ \hline 3 \end{array}$$

$\therefore$ the maximum relative error in the s. g.

$= \dfrac{{\cdot}0005}{24{\cdot}216} + \dfrac{{\cdot}0005}{11{\cdot}021}$

$= {\cdot}00002 + {\cdot}00005$ approximately

$= {\cdot}00007$ roughly.

$\therefore$ the maximum absolute error in the s.g.

$= {\cdot}00007\times\dfrac{24{\cdot}216}{11{\cdot}021}$

$= {\cdot}00014$ approximately.

The answer consists of 5 significant figures with a possible error in the fifth.

$\therefore$ the s. g. = 2·197.

Exercise VI (h).

The figures given are approximate only.

1. Find the weight of a block of metal 893·6... cubic centimetres, the weight of each cubic centimetre being 13·64...grams.

2. A room is 8·32 m. long, 6·48 metres broad and 2·98 high (each correct to a cm.) Find the volume as accurately as the data justify.

3. The length of a rod of iron is 32·253 ft. on a cold day. What will be its length on a day when each foot of the rod has expanded to a length of 1·000017 ft.?

4. What will be the yield in ounces of silver from 20, 864 tons 6 cwt. (to the nearest cwt.) of ore if each ton contains 8·936 lbs. of silver.

5. A gallon of fresh water contains 277·271.....cubic metres and weighs 10 lbs. avoirdupois. A ton of sea water measures 35 cubic feet to the nearest unit. Compare the weights of a gallon of fresh water and of a gallon of sea water. (Give your answer to the nearest trustworthy figure).

6. If 1 cubic foot of oxygen under certain circumstances of pressure and temperature weighs ·0893.....lb. and a cubic inch of mercury weighs 0·49.....lb., how many cubic inches of the oxygen would weigh as much as a cubic inch of mercury ? (Answer to be as accurate as the data justify).

7. Assuming that one metre = 39·37043 inches, what percentage of error is introduced (1) in linear measure ; (2) in square measure ; (3) in cubic measure ; by taking the decimetre as equivalent to 4 inches ?

8. Determine in the following questions (1) the maximum relative error of each factor to 2 significant figures; (2) the maximum absolute error of the product or quotient to 1 significant figure, it being assumed that each number is known correct to the places given :—

(*a*) 87·14 × 67·89.

(*b*) 12·1505 × ·00835.

(*c*) 64·768956 ÷ (i) 4·38; (ii) 4383 (iii) 4·3834.

9. If each number is known correct to the figures given, determine the number of significant figures to which we can correctly evaluate :

(*a*) $\frac{4{\cdot}896754321 \times {\cdot}00093876}{2{\cdot}398374569}$.

(*b*) $\frac{42{\cdot}98756432 \times 3{\cdot}08765912}{{\cdot}7654328943}$.

10. The diameter of the earth is 7912.....miles ; given that $\pi = 3{\cdot}1416$.....required the circumference of the earth at the equator and the number of miles per hour which the inhabitants of latitude 60° are carried by the earth's diurnal rotation.

§ 46. Decimalisation of English money.

In working problems, it is often found necessary to know how to readily express a given sum of shillings and pence, or of annas and pies, as the corresponding decimal of a pound or of a rupee as the case may be ; and also to readily read a given decimal of a £ or of a Rupee as shillings and pence, or annas and pies.

To begin with English money :

Since 20*s.* = £1

2*s.* = £0·1 ;

and 1*s.* = £0·05.

So, 14*s.* = £0·7,

and 17*s.* = £0·85.

So, to convert a given number of shillings into the decimal of a £, multiply such number by 0·05. Also, to read the decimal of a pound as shillings, divide the number formed by the first two digits after the decimal by 5, and set down the quotient as shillings.

Thus £0·378 = $\frac{3}{5}\frac{7}{}$*s.* + £0·008 = 7*s.* + £0·028.

Since 960 farthings = £1

1 farthing = £$\frac{1}{960}$ = £$\frac{1}{1000} \times \frac{1000}{960}$

= £0·001 $(1 + \frac{1}{24})$.

With this formula, a given number of pence may be readily expressed as the fraction of a £. Thus $8d. = 32$ farthings $= £0{\cdot}001\ (32 + \frac{32}{24}) = £0{\cdot}033$, approximately.

Hence, to convert pence into the decimal of a £, convert the pence into farthings; add to this number $\frac{1}{24}$ of it correct to an integer and divide the sum by 1000.

Also £1 = 960 farthings
$= 1000\ (1 - \frac{140}{1000})$ farthings
$= 1000\ (1 - \frac{1}{25})$ „

Hence, to convert the decimal of a pound to farthings, multiply the fraction by 1000; subtract from the resulting product $\frac{1}{25}$ of it correct to an integer; the answer is the number of farthings.

Example 1. Express 11*s.* 7*d.* as the fraction of a £.

11*s.* 7*d.* = 11*s.* 28 farthings = £(0·55 + 0·029) = £0·579 correct to 3 places of decimals.

Example 2. Express £2·736 as pounds, shillings and pence.

£2·736 = £(2 + ·70 + ·036) = £2 14*s.* + £0·036 = £2 14*s.* + 35 farthings = £2 14*s.* $8\frac{3}{4}d.$ to the nearest farthing; or £2 14*s.* 9*d.* to the nearest penny.

In the fraction of a pound, we need not recognise digits to the right of the 3rd place after the decimal point except to correct the third place; for if any sum be expressed as a fraction of a £ correct to 3 decimal places, the greatest possible error either way will be £·0005 = ·5 $(1 - \frac{1}{25})$ farthing which is less than half a farthing. So, any fraction of a pound correct to 3 places of decimals will be correct to a farthing.

§ 47. Decimalisation of Indian Money.

Now, 8 as. = half a rupee = 0·5 of a Re.
4 as. = quarter of a Re. = 0·25 of a Re.
2 as. = half of 4 as. = 0·125 of a Re.
So, 6 as. = (4+2) as. = 0·375 of a Re.
10 as. = (8 + 2) as. = 0·625 of a Re.
12 as. = (8 + 4) as. = 0·75 of a Re.
14 as. = Re. 1 minus 2 as. = 0·875 of a Re.

Again 1 pie $= \frac{1}{192}$ of a Re.
$= (\frac{1}{200} \times \frac{200}{192})$ of a Re.
$= \text{Re. } 0{\cdot}005\ (1 + \frac{8}{192})$
$= \text{Re. } 0{\cdot}001\ (5 + \frac{5}{24})$.

Thus 7 ps. $= \text{Re. } 0{\cdot}001\ (35 + \frac{35}{24})$ Re. 0·001 × 36 approximately.

(Here, $\frac{35}{24} = 1\frac{11}{24}$. We recognise 1 and neglect $\frac{11}{24}$ which is less than half.)

= Re. 0·036, correct to 3 places of decimals.

12 as. 7 ps. = Re. 0·75 + Re. 0·036
= Re. 0·786, correct to 3 places of decimals.

Again 11 as. 4 ps. may be converted thus: it = 10 as. + 16 ps.; 10 as. = Re. 0·625; and 16 ps. = Re. 0·001 $(80 + \frac{30}{24})$ = 0·083, correct to 3 places of decimals.

∴ 11 as. 4 ps. = Re. 0·625 + Re. 0·083 = Re. 0·708, correct to 3 places of decimals.

In converting pies into the decimal of a Re., this rule must be remembered: multiply the number of pies by 5 and add to this product $\frac{1}{24}$ of it; express this to the nearest integer, and divide by 1000.

Again Re. 1 = 192 pies
< 200 p.
∴ Re. ·0005 < ·0005 × 200 p.
< ·1 p.

∴ if a sum be expressed as the decimal of a rupee correct to 3 decimal places the greatest possible error either way will be only Rs. ·0005 which we have seen is less than ·1 p. ∴ any fraction of a rupee expressed correct to 3 decimal places will be correct to a pie.

For converting the decimal of a rupee into annas and pies, write it correct to the third decimal place and multiply by 16, the integral part gives the annas, then multiply the resulting fractional part by 12, the product corrected to an integer gives the pies.

Example 1. Convert 0·705 of a Re. into annas and pies.

Re. ·705 = ·705 × 16 as. = 11·28 as
= 11 a. + ·28 × 12 p.
= 11 a. 3 p.

Example 2. Express 7 as. 8 ps. as the decimal of a Rupee.

The answer is 6 as. + 20 ps. = Re. (0·375 + 0·104) = Re 0·479, correct to the thousandth.

Note.—Great care is required in dealing with the decimals of a Re. or a £. In adding several sums of money, each expressed correct to 3 places of decimals, the third place after the decimal point in the sum may not be correct, as it may be affected by the addition of figures in the 4th place after the decimal point if we had retained the 4th place.

Consider the product of Rs. 7-10-5 and 1000. Now Rs. 7-10-5 = (Rs. 7·625 + 0·026) approximately = Rs. 7·651. Then the product would be Rs. 7651. But multiplying Rs. 7-10-5 by 1000, in the ordinary way, the product will be Rs. 7651-0-8. In such calculations, as many places in

the fraction must be retained as will give the answer correct to 3 places of decimals.

Exercise VI (i).

Express the following quantities in 1—10 to the nearest pie or penny, as the case may be:

1. £16·347. **2.** Rs. 29·732. **3.** Re. 0·293.
4. Rs. 8·972. **5.** £15·017. **6.** Rs. 21·469.
7. £0·879. **8.** Rs. 11·677. **9.** £44·999.
10. Rs. 63·098.

Express the following quantities in 11—20 as decimals of a Rupee or a Pound as the case may be:

11. Rs. 27-5-7. **12.** Rs. 10-9-8. **13.** Rs. 5-15-11.
14. £17-11-5. **15.** Re. 0-6-10. **16.** £8-9-7.
17. Rs. 35-6-8. **18.** Rs. 9-9-9. **19.** Rs. 16-7-5.
20. £19-19-11.

21. A man recovers 35 per cent. of his debt. If the debt be £237 9*s.* 5*d.*, find in decimals the amount recovered, and write it correct to a penny.

22. If a sum of Rs. 245-11-4 be distributed equally among 100 persons, find, to the nearest pie, what each person gets.

23. An officer was appointed to a place on Rs. 225 per mensem. If he joined duty on the morning of the 9th July, find his salary for July, (correct to a pie).

24. Find, by practice, the value (correct to a pie) of 17 mds. 3 viss 29 seers of sugar at Rs. 2-5-4 per maund.

Exercise VI (j).

1. Show that $\frac{1}{(1\cdot008)^3} = 0\cdot976$ very nearly using the fact that $\frac{1}{(1+x)^3} = 1-3x$ when x is very small.

2. If $\pi^2 = 9\cdot877$, show that $\frac{1}{\pi^2} = \cdot101$.

3. Show that when x, y have small relative errors a, b respectively the error in $\frac{x^2}{y} = 2a+b$ very nearly.

4. From the observation of the eclipses of Jupiter's satellites it is found that the time taken by light to travel from the sun to the

earth is 499 seconds (with a possible error of 2 seconds either way). If the velocity of light is 186,330 miles per second, find the distance from the sun to the earth and the possible error in your result. (It will be sufficient to give your result and the possible error to the nearest hundred thousand miles.)

5. A distance is measured with a chain which is 22 yds. long. If the chain is 1 inch too short, find in inches the error in measuring 1 mile with this chain. To how many significant figures will the result given in yards be correct ?

6. The length of a piece of paper is 22·3 cm. and its width is 12·4 cm. If each dimension is correct to 0·1 cm., find the area as accurately as the given dimensions allow.

7. A bar of platinum increases 0·0000086 of its length when heated from 0°C to 1°C ; show that it increases $0{\cdot}0000086 \times 3$ of its volume.

8. The amount of rainfall at a certain station during a certain week is 3·48 inches (to the nearest hundredth of an inch) and this was 93 per cent. of the average (to the nearest integer). Find the average amount as accurately as the data allow.

9. The value of g is found from the formula $g = \frac{4\pi^2 l}{t^2}$ where l ft. is the length of a pendulum and t sec. the time of its oscillation. In an experiment $l = 9{\cdot}23$ correct to two places, $t = 3{\cdot}27$ correct to ·01 of a second ; find g as accurately as you can from the data, supposing the formula to be correct.

10. Find to the nearest pie the value of 98 yds. (correct to the nearest integer) of copper wire of which the section is circular of diameter ·32 in. (correct to ·01 in.), assuming that 1 cub. foot of copper weighs 8570 oz. correct to the units' place and that the value of 1 ton of copper is Rs. 1,200 which may be subject to an error of Rs. 30 either way.

11. Calculate the density of carbon dioxide from the following data as accurately as they justify, assuming the temperature and pressure to remain constant:—

Weight of globe full of air = 1896 grammes (correct to the nearest integer).

Weight of globe exhausted = 1866·5 grammes (correct to 0·1).

Weight of globe full of carbon dioxide = 1911·08 grammes (correct to 0·01).

12. From 107·447 grammes of silver nitrate 100 grammes of silver are obtained. If the weight of silver may be considered accurate and the weight of the salt may be 1% too great but cannot be too small, find the percentage of silver in silver nitrate, as accurately as the data give it.

13. π is very nearly 3·14159265.

If it is taken equal to $3+\dfrac{1}{7+\frac{1}{16}}$

find the error introduced. Show that the latter value of π is $\dfrac{355}{113}$ very nearly.

14. If 25·22 francs = £1, show that $£1 = 25+\dfrac{1}{4+\frac{1}{2}}$ francs and hence show that 227 francs = £9.

15. If 1 year = 365·2422 mean solar days, calculate the amount of relative error in each of the following cases :—

When 1 year = 365 days ;
1 year = 365·24 days ;
1 year = 365·25 days ;
1 year = 365·2425 days ;

and how many years should elapse to produce a difference of 1 day according to each assumption ?

16. The speed of a point on the equator of Mercury is 146·87 m. per sec. and that of a point on the equator of Venus is 454·58 m. Compare the speeds and give the answer in the form of a simple ratio.

17. During a storm the barometer at sea level stood as low as 27·466 inches. (This may be subject to an error of ·002.) Assuming that 1 cubic foot of mercury weighs 848·75 lbs. (correct to ·01 of a lb.), what was the pressure in lbs. per sq. inch ?

18. With slight interruptions the birth-rate of Prussia has fallen steadily from 42·6 per 1000 (correct to ·1) in 1876 to 30·7 per 1000 in 1910 (correct to ·1). Assuming that the number of births is the same in both the years 1876 and 1910, find by how much per cent. the population in 1910 increased as compared with that in 1876 as accurately as the data justify.

19. A hollow cube of iron is of uniform thickness. Its external length is 10 inches and its weight is 120 lbs. Find the volume of the internal cavity. The iron weighs 450 lbs. per cubic foot. If the length may be wrong by 0·1 in., the weight of the cube by an ounce and the weight of a cubic foot of the iron by 10 lbs., estimate the possible error in your result.

20. If a cube 10·25 inches in the edge expands so that each edge is lengthened in the ratio of 1·0038 to 1, what does the volume become (in cubic inches to one decimal place)? If any vessel expands so that every linear measurement is increased in the ratio of 1·0038 to 1, in what ratios are the area and the volume increased?

21. The length of a rectangular block is measured and found to be 7·8 cm., its breadth 4·5 cm., thickness 3·2 cm. and specific gravity is estimated 8·8. Find the limits to the possible weight of the block in grammes. (It should be assumed that there is no knowledge of the figures beyond those given but that the figures cited are correct to the last significant figure.)

22. In order to calculate the length of wire in a coil the following operations were performed :—

(1) The coil was weighed;
(2) a piece of wire was cut off and its length determined;
(3) the piece of wire was weighed.

If the weight of the coil was 1006·34 grammes, the piece of wire 98·5 cm. long and 1·06 grams in weight, find the limits of the length of the wire in the coil in cm.

23. Find the sum so that the error may be less than one-millionth of $1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2\,3.4}$.

24. The chemical composition of dry air is as follows :—

Nitrogen	...	78·39 per cm.
Oxygen	...	20·93 ,,
Argon	...	0·36 ,,
Carbon dioxide	...	0·32 ,,

Find the volume of nitrogen in a room 7·38...metres long 4·49... metres broad and 3·16...m. high. State your answer to the nearest cubic metre.

REVISION PAPERS.—I SERIES.

1.

1. If $x = 1, y = 2, z = 3$, find the value of $x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2$.

2. Factorize :—

(a) $x(x+y) + xy + y^2$.

(b) $(x+2y)^2 + 3x + 6y$.

3. Find in square feet the area of the kite shown in the adjoining figure on a scale of an inch to a foot.

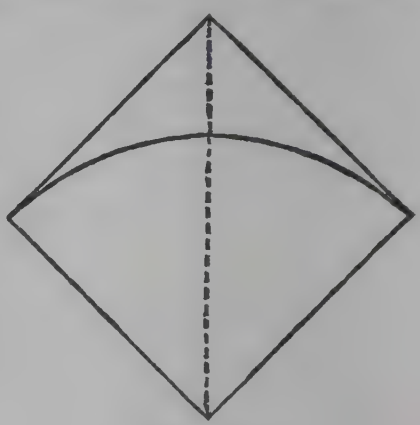

FIG. 38.

4. If 40 men can dig a ditch 900 yds. long, 8 ft. deep and 6 ft. broad in 72 days, how many men can dig a ditch of the same width, 300 yds. long, 6 ft. deep in 4 days ?

5. The sides of a rectangle are measured as a inches and b inches. Write down (1) the area of the rectangle; (2) the length measured round the rectangle; (3) the difference between the lengths of the sides. Give numerical values in each case taking a to be 56 and b to be 35.

6. If there is a possible error of x inches in each measurement so that one side of the rectangle may have any value between $(a+x)$ and $(a-x)$ inches and the other side any value between $(b+x)$ and $(b-x)$ inches, what is the greatest possible error in the area ?

Give the numerical value when $a = 56$, $b = 35$, $x = 0{\cdot}5$.

7. Solve :— $(x+a)(x+b) = (x+c)(x+d)$.

8. Pipe A fills a cistern in half an hour and pipe B in 40 minutes ; but owing to a crack in the bottom of the cistern it is found that pipe A now takes 40 minutes to fill the cistern. How long will the crack take to empty it and how long will it now take B to fill it ?

2.

1. Multiply $x^2 + xy - y^2$ by $x^2 - xy + y^2$ and verify your work by putting $x = 1$ and $y = -1$.

2. Two circles are described having their radii equal to 4·2 cm.

and 3·8 cm. respectively. Equilateral triangles are inscribed in them. Show that the sides of the equilateral triangles are in the ratio of the radii.

3. Prove that $\frac{1}{993} = \frac{1}{1000} + \frac{7}{(1000)^2}$ approximately and hence find the value of $\frac{8}{993}$ × Rs. 86754 to the nearest pie.

4. In Germany, Reaumur's thermometer is generally used, in France, the Centigrade; and in England, Fahrenheit's. Their freezing points are 0°, 0° and 32°, and their boiling points are 80°, 100° and 212° respectively. How would the Germans and the French respectively express a temperature of 98° Fahrenheit?

5. Assuming that a metre is equal to 39·37079 inches, express a square metre in square feet and decimals of a square foot correct to 4 significant figures.

6. Solve the following equation and verify the solution :—

$$\frac{5}{6}(3x - 11) - \frac{3}{4}(3x - 5) = \frac{x}{5} - (20 - x).$$

7. A and B go out for a holiday, A starting with Rs. 30 more than B. At the end of a week A has spent five-sixths of his money, B has spent a third of his and together they have Rs. 55 left. How much money did each start with?

8. Draw a graph to show the variations in the National Debt of England from the following data :—

Year.	1899	1901	1903	1905	1907	1908	1909
Debt correct to the nearest 10 millions.	£ 64	£ 70	£ 80	£ 80	£ 78	£ 76	£ 75

3.

1. Explain the application of the identity $(a + 1)^2 = a^2 + (2a + 1)$ as shown on the left-hand side to the construction of a table of squares and complete the table so as to show the squares of 1000 to 1005.

$1000^2 = 1000000$
$\quad\quad 2001$
$1001^2 = 1002001$
$\quad\quad 2003$
$1002^2 = 1004004$

2. If $a = 1005$, $b = 1001$, find the value of $2ab$ by using the identity $(a - b)^2 = a^2 - 2ab + b^2$.

3. Find, by drawing and calculation, the altitude of the sun when the shadow cast by a man is twice as long as himself.

4. The numbers given in the following table are known to follow approximately the law $y = a + bx$. Determine the probable values of a and b.

$y =$	·75	1·28	1·60	2·03	2·46
$x =$	1	2	3	4	5

5. The following statements contain obvious errors; show how to detect them with the least possible work. You are not to rectify them but to indicate why you consider them wrong.

(*a*) $\dfrac{29^3 \times 8 \times 24}{9 \times 23 \times 267} = 23{\cdot}08.$

(*b*) $(1{\cdot}38576)^2 = {\cdot}1798507.$

6. Multiply 3·14159 by 0·08732 so as to obtain the first four decimal places of the product.

7. The scale of a plan is to be reduced so that a portion which occupied 1·44 sq. in. may be reduced to 1 sq. in. If the length and breadth of the original plan be 12·8 in. and 10·4 in., find the length and breadth of the reduced plan.

8. A person bought 70 yds. of cloth for Rs. 26-9-0, some at 7 as. 6 pies a yard and the rest at 5 as. 6 pies a yard. How many yards of each kind did he buy? Check your answer graphically.

4.

1. (*a*) $(x+a)^2 = x^2 + 2xa + a^2$. Deduce from this the value of $(x-a)^2$.

(*b*) Prove that $(4 + 5)3 = 4 \times 3 + 5 \times 3$. Name the law of which this equality is an instance.

2. A wooden ball 75 cm. in diameter weighs 47·2 kg. Find to the nearest hundredth of a gram the weight of a cubic centimetre of wood. The volume of a ball of diameter d is $\frac{1}{6}\pi d^3$ where $\pi = 3{\cdot}1416$.

3. Find the area of the field drawn in the adjoining figure to a scale 1 inch to 240 yds. and from the result of your calculation or

otherwise draw a line parallel to AB that will divide the figure into two equal parts.

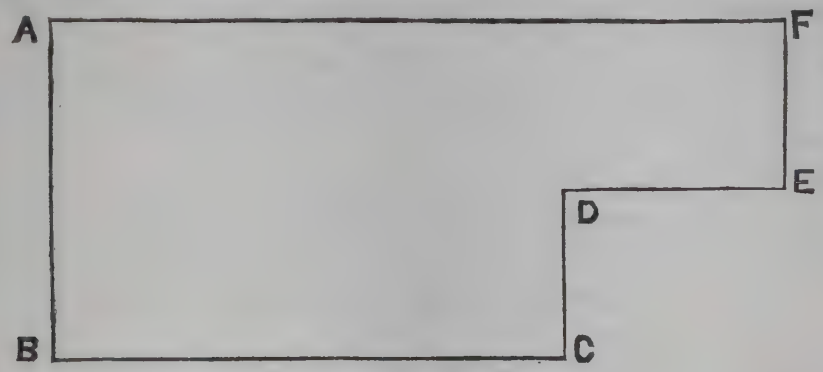

FIG. 39.

4. Factorise :—

(*a*) $7x^2y - 28y^3$.

(*b*) $a^2 - 9b^2 - 30b - 25$.

5. 3000 workmen strike because an advance of one half anna per hour is not conceded. Find the additional sum which would be required weekly to concede the demand, taking a working week to consist of 6 days and a working day to consist of 9 hours.

6. A man 5 ft. 10 inches in height standing at a distance of 54 ft. from the base of an electric light tower casts a shadow 8 ft. long on the pavement. What is the height of the tower?

7. Solve :—

$$\frac{(x-3)(x-4)}{3} = \frac{(2x-3)(6x+5)}{36} + \frac{8}{15}.$$

8. An officer forms his men into a hollow square 6 deep. If he has 2256 men, find how many men there will be in the front.

5.

1. Illustrate the following identities graphically :—

1. $(x+3)(x+5) = x^2+8x+15$.
2. $(x-4)^2 = x^2 - 8x + 16$.

2. The product of 2071723 and 5363222357 is

11111111111111111.

Hence find the product of 2071724 and 5363222356.

3. Water runs at $5\frac{1}{2}$ miles an hour along a pipe the cross section of which is 4 sq. ft. How many gallons (to the nearest million) does it deliver in 24 hours ? (A cubic foot is 6·23 gallons.)

4. From the following table of co-efficients of linear expansion for 1°C., construct a table for the co-efficients of cubical expansion :—

Glass, Platinum	= 0·0000085.
Cast iron	= 0·00001.
Copper	= 0·000017.
Lead	= 0·000028.
Zinc	= 0·00003.

5. If a water pipe is p yards long, d inches in diameter, and one end is h ft. higher than the other, then $\sqrt{(3d)^5 \times h \div p}$ gallons of water flow through the pipe in one minute. Use this formula to find how many gallons per minute to the nearest unit will flow through a pipe 1 mile long, 4 in. in diameter and one end being 30 ft. higher than the other.

6. Determine the yearly incomes each an exact number of rupees, such that the corresponding average daily incomes lie between Rs. 3-15-4 and Rs. 3-15-11 (1 year = 365 days).

7. A litre of water weighs a kilogram, a litre of another liquid weighs 1·480 kg. A mixture of the two weighs 1·280 kg. per litre. Determine the volume of each in a litre of the mixture.

8. Solve $\frac{x-5}{3} = \frac{6x-8}{3}$.

Put $\frac{x-5}{3} = y$ and draw the graph. Similarly put $\frac{6x-8}{3} = y$ and draw the graph. Find out the abscissa of the point of intersection of these two graphs and show that the value is the same as the root of the equation.

6.

1. What is the error involved in assuming that 3·14159265 $= \frac{3}{1}\frac{5}{1}\frac{8}{3}$? Which is the greater of these two quantities ? Give your result to 9 places of decimals.

2. A pendulum of length l ft. is known to make a complete swing in $2\pi\sqrt{\frac{l}{g}}$ seconds, where $\pi = \frac{22}{7}$, $g = 32$. Find the time of a complete swing for a pendulum 100 ft. long.

3. Jupiter rotates in 9 hours 55 minutes 20 seconds. If its diameter be 88200 miles, find to one decimal place the number of miles which a particle on its equator travels in one second in consequence of its rotation. The circumference of a circle may be taken as 3·1416 times its diameter.

4. The watch used to time a cycle race gained x seconds a day. The time of the race as given by the watch was a min. b sec. Find an expression in seconds for the true time of the race. If the watch lost x seconds, how would you change your answer ?

5. Three ships A, B and C at anchor occupy the following positions with regard to a buoy Q :—

A is 400 yds. north of Q ; B 180 yds. north-east of Q ; C is 360 yds. east of Q. Draw a plan showing the positions of the ships and the buoy on a scale of 100 yds. to an inch. A fourth ship D anchors in a position equidistant from A, B and C. Determine its position in your plan stating its distance from the buoy.

6. Factorise :—

(*a*) $x^2 - 2xy - 15y^2$.

(*b*) $250 - 16x^3$.

(*c*) $(a+b)^2 - 7(a^2 - b^2) + 10(a-b)^2$.

7. A square lawn is surrounded by a path 8 ft. wide ; the area of the path is 1024 sq. ft. Find the area of the lawn.

8. Solve and verify the solution :—

$(x+3)(x+5)(x-2) = (x+9)(x+8)(x-11)$.

7.

1. By means of factors, find the product of $(2a^2 - 7ab + 5b^2)$ and $(2a^2 - 7ab - 5b^2)$.

2. Find the absolute error in reckoning the area of a zinc square 1 foot side at 0°C. equal to 1·00006 sq. ft. at 1°C. Use the table of co-efficients of linear expansion given in paper **5** of the series. Find the relative error.

3. The driving wheel of a locomotive engine 5 ft. 6 in. in diameter turned 2400 times in going 8 miles. Supposing the circumference of a circle to be 3·1416 times the diameter, find (to the nearest yard) what distance is lost owing to the slipping of the wheel on the rail.

4. The scale of a map is 1 : 12500000. Find in kilometres the real distance represented by a centimetre on the map ; and express as the decimal of an inch the distance on the map which represents a mile.

5. A train which travels at the rate of 48 miles an hour leaves P at noon, meets another train from Q at 2-30 P.M., and arrives at Q at 3-50 P.M. At what rate does the second train travel if it left Q at 12-25 P.M. ?

6. Evaluate graphically the following quantities as accurately as you can :—

(i) $\frac{21{\cdot}3}{68{\cdot}9} \times \frac{21{\cdot}3}{68{\cdot}9} \times 5{\cdot}21.$

(ii) $\frac{21{\cdot}3}{68{\cdot}9} \times 5{\cdot}84.$

7. The following table, taken from an actual case, gives the cost of a book per copy, according to the number printed. Draw a graph giving the cost per copy, according to the number of copies. Find from it how many copies you must print so that each copy may cost less than 12 as.

Number of copies.	1000	2000	3000	4000
Cost per copy.	14·52 as.	12·38 as,	8·76 as.	7·64 as.

8. Assuming that the price of a diamond per carat varies as the square of its weight and a diamond of 12 carats costs Rs. 5000, find the cost of a diamond of 4 carats.

8.

1. The ratio of the height of an equilateral triangle to a side is 0·8660254. Show that ·8660254

$$= \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{2+\frac{1}{8}}}} = \frac{110}{127} \text{ nearly.}$$

To how many significant figures is this correct?

2. Find the maximum relative error in $(\frac{1}{9\cdot148})^2$ if the denominator is correct to four figures.

3. Evaluate $\frac{8\cdot3624 \times 0\cdot0005268}{(0\cdot03287)^2}$ to 3 significant figures.

4. A train 66 yds. long moving at a uniform rate overtakes a man who is walking along the line at the rate of 4 miles an hour and passes him in 6 seconds. Find the velocity of the train. Twenty minutes after it has passed him it meets a man walking in the opposite direction and passes him in 4½ seconds. How long after this will the two men meet?

5. In what ratio must two teas costing Rs. 150 and Rs. 100 per cwt. respectively be mixed, in order that a profit of 10 per cent. may be made by selling the mixture at Re. 1-6-0 a lb?

6. Draw the quadrilateral ABCD, the corners A, B, C, D having the co-ordinates, (14,12), (12,2) (4,11), (1,15). Suppose this represents a field on a scale 1 to 1440, find the area of the field in acres.

7. Solve:—

$$\frac{x+a+c}{x+b+c} = \frac{b}{a}.$$

8. A crew which can row at the rate of 10 miles per hour on still water finds that it takes twice as long to pull up a certain reach against stream as it does to come down the same reach Find the rate of the stream.

9.

1. If $\pi = 3\cdot14159265$, find the value of π^2 and π^3 to four places of decimals and also $\frac{1}{\pi^2}$ and $\frac{1}{\pi^3}$ to 4 places of decimals.

2. Express the weight of a cubic metre of water in decimals of a ton, having given that a cubic foot of water weighs 62·5 lbs., and that the circumference of the earth is 40000000 metres or 25000 miles.

3. Obtain Rs. $23{\cdot}678 \times (1{\cdot}025)^4$ correct to the nearest anna.

4. The weights of three diamonds are to one another as 3 : 5 :7. The weight of a fourth diamond is $1\frac{1}{3}$ times the total weight of the three diamonds. Assuming that the values of the four diamonds are proportional to the squares of their respective weights and that the heaviest diamond costs Rs. 1500, find the values of the other diamonds to the nearest rupee.

5. What is the error per cent. in the assumption $(1+a)(1+b) = 1+a+b$, if $a = {\cdot}0002$, $b = {\cdot}005$?

6. Cycling at the same time every morning at 12 miles an hour, I generally pass at the same spot a man walking in the same direction at $4\frac{4}{5}$ miles an hour. One morning being 4 minutes late, I pass the same man a mile beyond the usual spot. Is he late or early and by how many minutes? If I had been punctual where should I have passed him?

7. By means of the formula $a^2 - b^2 = (a+b)(a-b)$, find the value of $4234 \times 3766 - 2226 \times 1774$.

8. Draw a graph to show the fall of the barometer due to increase of height above the sea level from the following data.:—

Height in thousands of feet	0	5	10	15	20	25	30
Height of barometer in inches	30	24·8	20·5	17·0	13·9	10·9	9·5

Estimate the height of the barometer at 17000 ft, above sea level.

10.

1. If there is a possible error of 1 $^0/_0$ in my measurements show that the area of a rectangle deduced from them may be as much as 2 $^0/_0$ in error and that the volume of a cuboid may be as much as 3 $^0/_0$ in error.

2. Divide $3x^3 - 13x^2 + 23x - 21$ by $3x - 7$.

3. Using the value $(1 + a)^n = 1 + na$, find the value of $(\cdot 997)^3$ and find the error per cent.

4. The height of a mountain is given in a map in metres and I want to turn it into feet. If I reckon 11 metres = 12 yds., I get the answer too small by 40 ft. If I reckon 8 kilometres = 5 miles, I get the answer too big by 92 feet. From these data calculate to two places of decimals the number of inches in a metre.

5. One train leaves A at 12 noon and reaches B(120 m. from A) at 2 P.M.; another train leaves B at 12-30 P.M. and reaches A at 2·45. When and where would you expect them to meet? Illustrate graphically.

6. Factorise as fully as possible :—

1. $2a - 18a^3$.
2. $6a^2 - ab - 2b^2$.
3. $(a^2+b^2)^2 - 4a^2b^2$.

7. There are two mixtures of wine and water, one of which contains as much water as wine, and the other four times as much wine as water. How much must be taken from each to fill a a pint cup, in which the water and wine shall be in the ratio ~~2 : 3~~ 2 : 3

8. Solve :—

$$(2x-3)(3x-5)+(4x-3)(3x-7) = (3x-6)^2+(4x-1)(2x-7)+(x+3)(x-3).$$

CHAPTER VII.

QUADRILATERALS AND CIRCLES.

§ **48. A quadrilateral** is a four-sided figure.

Exp. Draw any quadrilateral ABCD, measure the angles and find their sum. Repeat the experiment with two or three quadrilaterals. What do you conclude? You will find that **the angles of a quadrilateral are together equal to four right angles.**

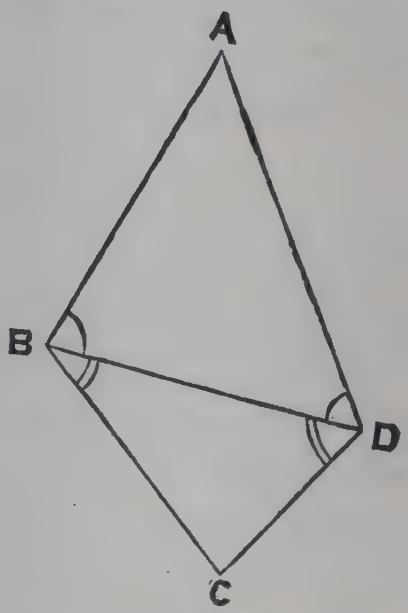

FIG. 40.

The same may be proved thus:—Let ABCD be the quadrilateral. Join BD. Then ABD is a triangle. $\therefore \angle A + \angle ABD + \angle ADB = 2$ right angles. Similarly in the $\triangle CBD$, $\angle C + \angle CBD + \angle CDB = 2$ right angles.

$\therefore$ the angles of these two triangles taken together are equal to four right angles, *i.e.*, $\angle A + \angle C + (\angle ABD + \angle CBD) + (\angle ADB + \angle BDC) = 4$ rt. angles,

i.e., $\angle A + \angle C + \angle B + \angle D = 4$ rt. angles.

The same thing may be proved by taking a point within the quadrilateral and joining it to each of the angular points as in fig. 41. Then the quadrilateral is divided into 4 triangles.

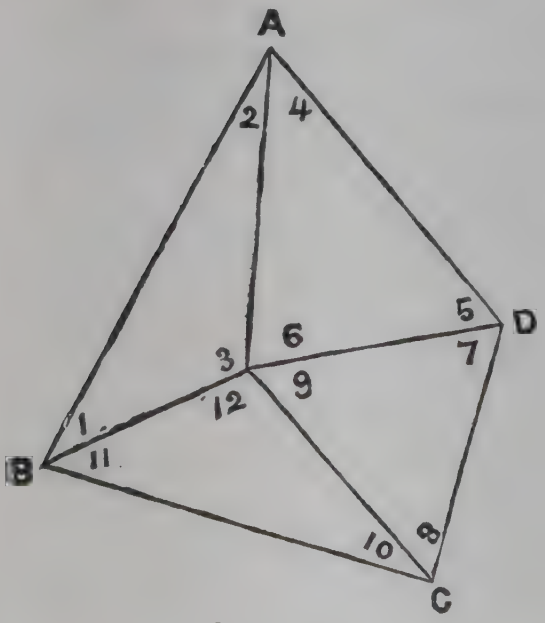

FIG. 41

Then we have

1 + 3 + 2 = 2 rt.∠s.
4 + 6 + 5 = do.
7 + 9 + 8 = do.
10 + 12 + 11 = do.

$\therefore$ $1+3+\overline{2+4}+6+\overline{5+7}+9+\overline{8+10}+12+11 = 4\times 2$ rt. angles,

i.e, $3+6+9+12+\angle A+\angle D+\angle C+\angle B = 4\times 2$ rt. angles,

for $\overline{2+4} = \angle A$ in the figure

$\overline{5+7} = \angle D$,,

$\overline{8+10} = \angle C$,,

$\overline{11+1} = \angle B$,,

But $3+6+9+12 = 4$ rt. angles, for any number of angles meeting at a point are together equal to 4 rt. angles.

$\therefore$ 4 rt. angles + $\angle A$ + $\angle B$ + $\angle C$ + $\angle D$ = 4 × 2 rt. angles. ... ... (A)

$\therefore$ $\angle A + \angle B + \angle C + \angle D = 4 \times 2 - 4$ rt. angles, *i.e.*, 4 right angles.

Note 1.—If instead of a quadrilateral, a **pentagon** (a five-sided figure) ABCDE is taken, by taking a point inside and joining it to each of the angular points it may be shown as in (A) that 4 right angles + $\angle A$ + $\angle B$ + $\angle C$

$+ \angle D + \angle E = 5 \times 2$ rt. angles; a similar result can be obtained for a six-sided figure, **(a hexagon)**, a seven-sided figure **(a heptagon)** and in general for a figure of n sides in which case we get 4 right angles $+ \angle A + \angle B + \ldots\ldots + \angle X$ (n angles) $= n \times 2$ rt. angles, *i.e.*, **all the angles of a rectilineal figure together with 4 rt. angles are equal to twice as many right angles as the figure has sides.**

Note 2.—4 rt. angles + the n angles of the figure $= 2n$ rt. angles. $\therefore$ the n angles $= (2n - 4)$ rt. angles.

Suppose that all the n angles are equal to one another, *i.e.*, the figure is equiangular.

Then each angle $= \frac{2n - 4}{n}$ or $\left(2 - \frac{4}{n}\right)$ rt. angles

$$= \left(2 - \frac{4}{n}\right) 90^\circ = 180^\circ - \frac{360^\circ}{n}.$$

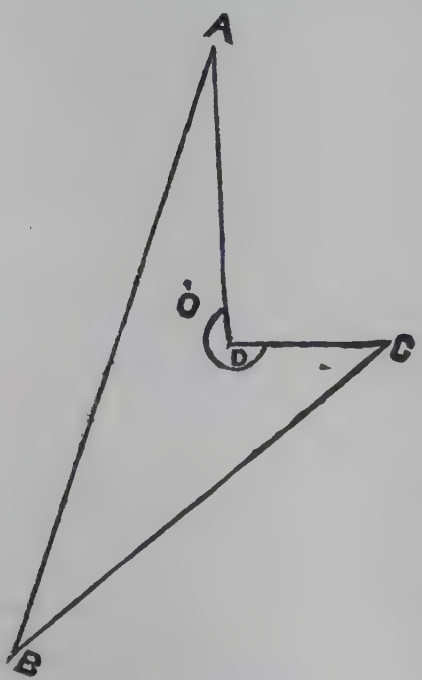

FIG. 42.

Note 3.—The polygon must be such that all the lines drawn from O, wherever O be taken within the figure, to the angular points fall within the figure. It is clear in a quadrilateral of the kind given in fig. 42, some of the lines drawn from O to the angular points will fall outside. This is so because the $\angle D$ is greater than 2 rt. angles; such an angle is called a **reflex angle**; a polygon in which there is no reflex angle is called a **convex polygon.**

Exercise VII (a).

1. What is the magnitude of all the angles of (1) a hexagon, (2) an octagon, (3) a decagon, and (4) a dodecagon.

2. Supposing the above figures are regular figures, what is the magnitude of each of the angles of the different figures ?

Definition.— A regular polygon is a figure bounded by straight lines having all its sides and angles equal.

3. Three angles of a triangle are collectively equal to the three angles of another triangle. Is the same true in any other rectilineal figure? Give reasons for your answer.

4. Show by dividing a pentagon into three triangles by joining any one corner to the other corners, that the angles are together equal to the same number of right angles as you get from Note 1.

5. Into how many triangles can a hexagon be divided in a similar manner and what inference can you then draw about the angles of the hexagon ?

6. Into how many triangles can a figure of n sides be similarly divided ? From a consideration of the angles of these triangles show that the angles of a figure of n sides are equal to $2(n-2)$ right angles.

7. If an angle of a regular polygon $= 135^0$, find the number of sides in the polygon.

8. If an angle of a regular polygon $= \frac{26}{15}$ of a right angle, find the number of sides.

§ 49. Exterior angles of a polygon.

ABCDE is a convex polygon. Let the sides be produced in order in the same sense. At every angular point of the polygon we then have an interior angle and an exterior angle which are together equal to two right angles and if there are n angular points, the n interior angles + the n exterior angles are equal to $n \times 2$ right angles.

But the n interior angles + 4 right angles have been proved to be equal to $n \times 2$ right angles.

$\therefore$ the n interior angles + the n exterior angles = the n interior angles + 4 right angles.

$\therefore$ the n exterior angles = 4 right angles.

Hence "**If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is equal to four right angles.**"

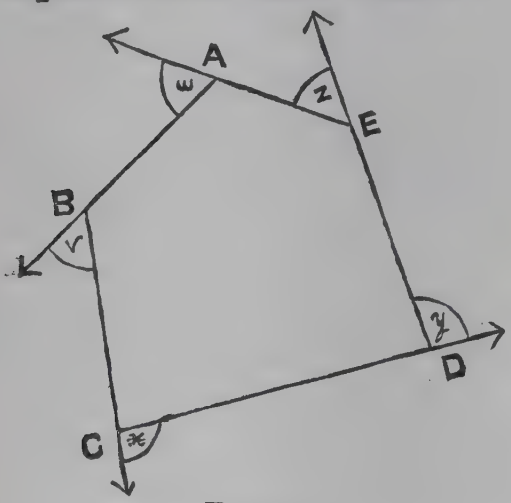

FIG. 43,

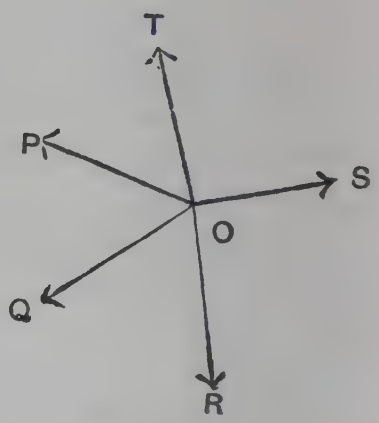

FIG. 44.

2nd method. The same thing may be proved in the following manner:

Take any point O; draw OP, OQ, OR, OS, OT $\parallel$ to and in the same sense as EA, AB, BC, CD, DE respectively (see figs. 43 and 44).

$\because$ OP, OQ are respectively parallel to and drawn in the same sense as EA, AB

$$\therefore w = \angle POQ$$

Similarly $\angle v = \angle QOR$

$\angle x = \angle ROS$

$\angle y = \angle SOT$

$\angle z = \angle TOP.$

$\therefore \angle w + \angle v + \angle x + \angle y + \angle z =$ sum of the $\angle$s at O
$= 4$ right angles.

3rd method. Imagine a rod placed in the direction of CD. Turn it about D in the counter-clockwise direction until it comes to the position DE. In so doing it describes an angle $= y$; next turn it about E until it comes to the position EA; in so doing it describes a further angle equal to z; again turn it about A until it comes to the position AB, and so on until it comes back to the position CD. You will then find that it has made a complete revolution.

$\therefore$ the angles y, z, w, v, x are together equal to 4 right angles.

Exercise VII (b).

1. What is the size of each exterior angle of a regular (1) pentagon, (2) hexagon, (3) octagon and (4) decagon?

2. Hence find the value of each of the interior angles of the regular figures given in Ex. (1).

3. An exterior angle of a regular polygon is 60°. Find the number of sides of the polygon.

4. How many sides have the regular polygons whose exterior angles are respectively 10°, 5°, 1° and $2\frac{1}{2}$°?

5. Is it possible to have regular polygons whose exterior angles are 7°, 11°, 13°, 6°, 5°, 4°?

6. Is it possible to have regular polygons whose interior angles are 108°, 120°, 130°, 144°, 60°?

7. Make a table showing the magnitudes of interior and exterior angles of regular polygons of 3, 4, 5,.....10 sides. Draw a graph for (1) interior angles (2) exterior angles.

8. Construct a regular octagon having each side 1 cm.

§ 50. Constructions of Quadrilaterals from given data.—As in the case of triangles, the

construction of quadrilaterals is possible when a certain number of the elements of a quadrilateral are given. We have seen that at least three elements of a triangle are necessary for the construction of a triangle and that sometimes no construction is possible when the three given elements happen to be of a particular kind, *viz.*, 3 angles. Seeing that a quadrilateral is made up of two triangles one may expect that a quadrilateral may be constructed if six elements are given. It is not so; for the two triangles have a side common, (*viz.*, the diagonal dividing the quadrilateral into two triangles), and hence, in general, *five* elements are enough to determine a quadrilateral, but a suitable set of five elements should be given and not any set.

There are some cases in the construction of quadrilaterals which involve a knowledge of properties of circles which have not been learnt till now. In the present chapter we shall only deal with some simple cases.

Ex. 1. Construct a quadrilateral ABCD given that AB = 3·1 cm., $\angle B = 80°$, BC = 4·1 cm, $\angle C = 70°$, CD = 4 cm. It is specially important that before beginning the construction, a rough sketch should be made showing the given parts.

Note.—The letters must be taken in order round the quadrilateral.

Construction. Draw a straight line AB = 3·1 cm. with your scale. At B make an angle of 80° with your protractor. Make BC = 4·1 cm. At C make an angle of 70°. Measure CD = 4 cm. Join AD. Then ABCD is the quadrilateral required.

Exercise VII (c).

Construct quadrilaterals to the following measurements :—

1. AB = 3 in., BC = 2 in., AD = 2·5 in. $\angle A = 65°$, $\angle B = 85°$.

2. AB=1·8 in., BD=1·6 in., CD=1·9 in. ∠ABD=110°, ∠BDC=25°.

3. AB=CD=5 cm., AD=4·2 cm., ∠A=75°, BDC=45°.

4. AB=5 cm., AC=6 cm., AD=5·5 cm., ∠BAC=105° ∠BAD=120°.

Ex. 2. Construct a quadrilateral ABCD, ∠A = 60°, ∠B = 110°, ∠D = 70°, AB = 4·8 cm., AD = 6·3 cm.

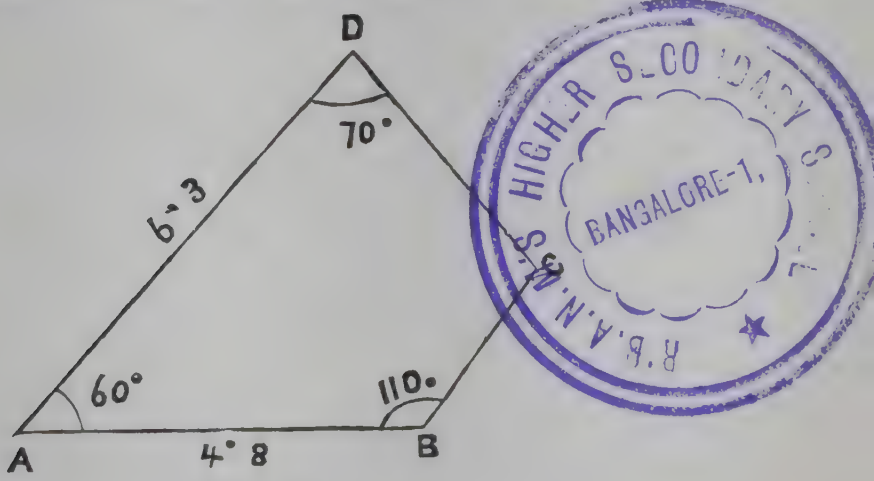

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FIG. 45.

Construction. Draw AB equal to 4·8 cm. At A and B make angles BAD and ABC equal to 60° and 110° respectively. Make AD = 6·3 cm. At D make an angle ADC = 70°. DC intersecting BC at C. This is the quadrilateral required.

Exercise VII (d).

Construct quadrilaterals to the following measurements :—

1. ∠B=120°, ∠C=125°, ∠D=∠125°, BC=CD=1·5 in.

2. ∠B=110°, ∠C=90°, ∠D=70°, AB=1·3 in., BC=1·9 in.

It is not possible to give all cases here. The student can in a similar manner proceed to construct quadrilaterals in the following examples.

3. AD=2 in., ∠D=112°, ∠DAC=30°, ∠BAC=32°, AB=2·5 in.

4. AC = 5·2 cm., ∠BAC = 56°, ∠DAC = 65°, ∠BCA = 57°, ∠DCA = 70°.

5. AB = 1·8 in., BD = 1·6 in., CD = 2·1 in. ∠ABD = 110°, ∠BDC = 25°.

6. AB = CD = 5 cm., AD = 4·5 cm., ∠A = 72°, ∠BDC = 50°.

7. AB = 6 cm., CD = 5·1 cm., ∠BAC = 60°, ∠ACD = 55°, ∠D = 75°.

8. AB = 5 cm., AC = 6 cm., AD = 5 cm., ∠BAC = 105°, ∠BAD = 120°.

9. AB = 3cm., BC = 4 cm., ∠ABC = 50°, ∠CAD = ∠ACD = 35°.

10. ∠BAC = 30°, AC = 4 cm., AB = 2·1 cm., ∠ABD = 35°, BD = 4·2 cm.

11. AB = 4·2 cm., AC = 5·6 cm., BC = 2·2 cm., ∠CAD = 30°, ∠ACD = 40°.

12. AB = 2·1 in., BC = 2·4 in., AC = 3·2 in., BD = 1·5 in., DC = 2·3 in.

§ **51.** The magnitude of an angle of a regular polygon can be inferred from the following properties of the circle.

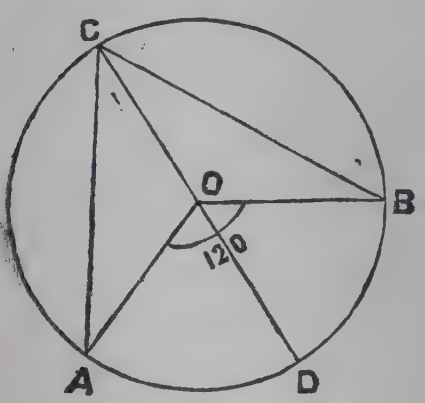


FIG. 46.

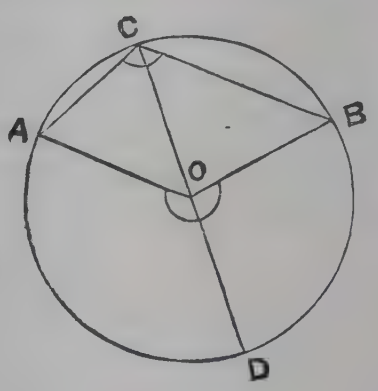


FIG. 47.

Exp. 1. Draw a circle ABC with O as centre.

At O, construct any angle AOB, say, of 120°. Then the circumference is divided into two parts, *viz.*, ADB and ACB as in Fig. (46). Take any point C in the segment

ACB join CA . CB and measure the angle ACB. You will find it to be 60° ; take other points in the arc of the segment ACB and measure the angles formed by joining each to A and B ; then each of the angles in the segment ACB, *i.e.*, standing on the arc ADB will be found to be 60°.

Exp. 2. At O, make an angle AOB of 100°. Measure the angles in the segment ACB. You will find that each of the angles in the segment ACB or on the arc ADB is 50°.

These experiments teach us (1) that angles in the same segment (or on the same arc) of a circle are equal to one another ; (2) that the angle at the centre of a circle is double the angle at the circumference standing on the same arc.

Besides the experimental verification, a proof can be given for the results (2) and (1).

In fig. 46 or 47 join CO and produce it to D.

$\angle AOD = \angle ACO + \angle CAO$

$= 2\angle ACO$. for $\angle ACO = \angle CAO$ $\because$ $OC = OA$.

Similarly $\angle DOB = 2\angle OCB$.

$\therefore$ $\angle AOD + \angle DOB = 2(\angle ACO + \angle OCB)$,

i.e., $\angle AOB = 2\angle ACB$.

i.e. The angle at the centre of a circle is double an angle at the circumference standing on the same arc.

N.B. From Fig. 47, it will be seen that this proposition is true even if the angle at the centre is greater than two right angles. (An angle greater than two right angles is called a **reflex angle.**)

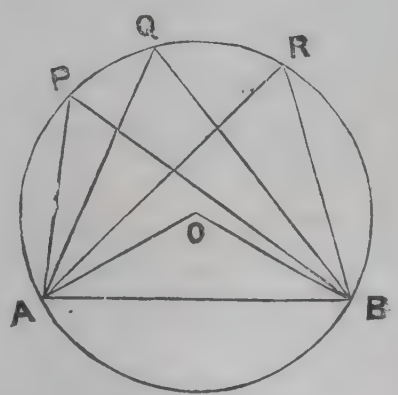


FIG. 48.

Result (1) may be proved thus:

Let the angles APB, AQB, and ARB stand on the same arc ADB.

$\angle APB = \frac{1}{2}\angle AOB$ standing on the same arc.

Similarly $\angle AQB = \frac{1}{2}\angle AOB$ and $\angle ARB = \frac{1}{2}\angle AOB$.

$\therefore \angle APB = \angle AQB = \angle ARB$.

i.e., angles in the same segment of a circle are equal.

Conversely, the vertices of all triangles standing on the same base and on the same side of it and having their vertical angles equal lie on an arc of a circle passing through the extremities of the base.

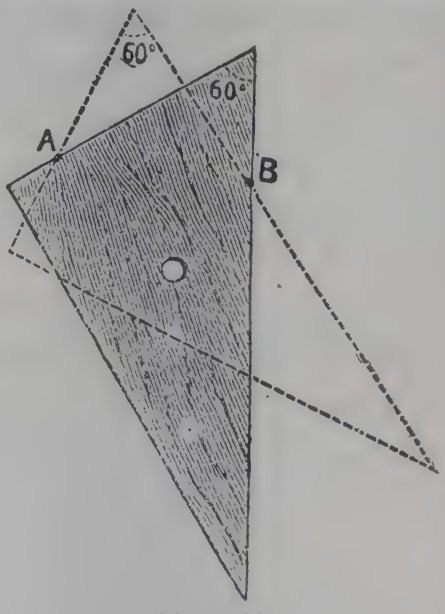


FIG. 49.

Mark two points A and B on your paper and place your set square so that two of its edges pass through A and B. Mark on paper the position of the vertex. You will find that there are a great many positions for it. Mark a large number of them and freely join them. You will find you get a segment of a circle.

An indirect method of proof of this converse may be given.

Exercise VII (e).

1. ABC is an equilateral △ inscribed in a circle of which O is the centre. Find the magnitude of each of the angles, OAB, OBC, and OCA.

2. ABC is an isosceles triangle on the base BC, inscribed in a circle of which O is the centre. Given that the angle BOC is 72°, find the angles of the isosceles triangle.

3. ABCD is a square inscribed in a circle. P is any point in the arc AB, Q in BC, R in CD. Calculate the magnitude of the angles (1) APD ; (2) AQD ; (3) ARD and verify by measurement.

4. In the same figure find the magnitude of each of the angles (1) BPD ; (2) BQD ; (3) BRD.

5. ABCDE is a regular pentagon inscribed in a circle of which the centre is O. What is the magnitude of each of the angles AOB, BOC, COD, DOE, EOA? From the magnitude of these angles taken in combination, find the value of the angles A, B, C, D and E.

6. As in 5, find the magnitude of an angle of (1) a regular hexagon, (2) a regular octagon and (3) a regular polygon of n sides.

§ **52.** Describe a circle with O as centre and in it inscribe a quadrilateral ABCD. Measure the opposite angles BAD and BCD and find their sum. Similarly find the sum of the opposite angles ABC and ADC. Repeat the experiment with different circles inscribing different quadrilaterals in them. You will find that the sum of the opposite angles of such quadrilaterals is two right angles.

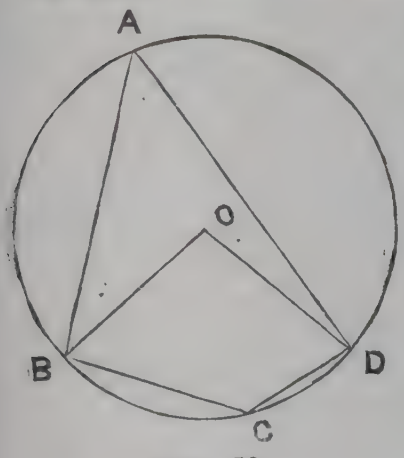


FIG. 50.

That it is so, can be proved thus :

$\angle BAD = \frac{1}{2} \angle BOD$ at the centre standing on the same arc BCD.

$\angle BCD = \frac{1}{2}$ reflex $\angle BOD$ standing on the same arc BAD.

$\therefore \angle BAD + \angle BCD$

$= \frac{1}{2}$ (∠BOD + reflex ∠BOD)

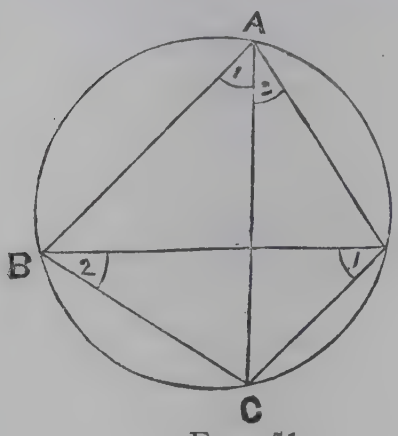


FIG. 51.

$= \frac{1}{2}$ of 4 right angles
= 2 right angles.

Or thus:

Join BD and AC. (Fig. 51)

∠BAC = ∠BDC being angles in the same segment BADC.

∠CAD=∠CBD being angles in the same segment CBAD.

∴ ∠BAC +∠CAD=∠BDC + ∠CBD,

i.e., ∠BAD = ∠BDC + ∠CBD.

To each of these add the ∠BCD.

∴ ∠BAD + ∠BCD = ∠BDC + ∠CBD + ∠BCD = 2 rt. angles (∵ these three are the angles of the △BCD).

Thus we see that **the sum of the opposite angles of a quadrilateral inscribed in a circle is two right angles.**

Conversely. If a pair of opposite angles of a quadrilateral are together equal to two right angles, the quadrilateral is **cyclic**, *i.e.*, a circle can be described so as to pass through its angular points. (The proof of this is left as an exercise to the student).

Exercise VII (f).

1. ABCD is a cyclic quadrilateral. ∠ABC = 60°, find the magnitude of the angle ADC.

2. In the same figure draw another angle AEC in the segment ADC. From the quadrilateral AECB calculate the magnitude of the angle AEC. Hence show that angles in the same segment of a circle are equal.

3. In the same figure produce AD to F. Show that $\angle CDF = \angle ABC$.

4. ABCD is a cyclic quadrilateral. $\angle ABC = 80^\circ$, and is bisected by BE meeting the arc AEC, in the point E. What is the magnitude of the angle EDC?

5. ABCD is a cyclic quadrilateral. AD and BC produced meet at P. Show that $\triangle$s PDC and PAB are equiangular.

§ **53.** Let ACB be a semi-circle of which AB is the diameter and O the centre. Take a number of angles in the semi-circle ACB; by measurement it will be found that each of these angles is a right angle. The same thing may be proved as follows:—

$\angle ACB = \frac{1}{2} \angle AOB$ (straight angle) at the centre standing on the same arc AKB $= \frac{1}{2} \times 2$ rt. angles $= 1$ rt. angle.

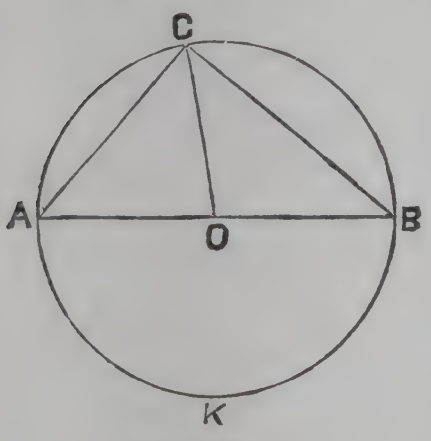


FIG. 52.

Or thus: Join OC.

In the $\triangle$ OCA, OA = OC.

$\therefore \angle OCA = \angle OAC$.

Similarly in the $\triangle$ OCB, $\angle OCB = \angle OBC$.

$\therefore \angle OCA + \angle OCB = \angle OAC + \angle OBC$,

i.e., $\angle ACB = \angle OAC + \angle OBC$.

$\therefore \angle ACB + \angle CAB + \angle CBA = 2\angle ACB$,

But $\angle ACB + \angle CAB + \angle CBA = 2$ rt. angles.

$\therefore 2\angle ACB = 2$ rt. angles.

$\therefore \angle ACB = 1$ rt. angle.

$\therefore$ **the angle in a semi-circle is a right angle.**

Let ACB be a segment greater than a semi-circle as in Fig. 53. Take a number of angles in this segment; by measurement, it will be found that each of these angles is an acute angle.

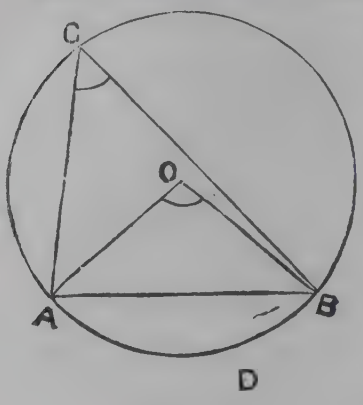


FIG. 53.

The same may be proved thus :— Since the segment is greater than a semi-circle the centre O should fall within the segment. Join AO, OB. The arc ACB is greater than the semi-circumference. ∴ the arc ADB is less than semi-circumference. ∴ ∠AOB is less than 2 rt. angles. ∴ ∠ACB being $\frac{1}{2}$ ∠AOB is less than $\frac{1}{2}$ of 2 rt. angles, *i.e.*, less than *1 right angle.*

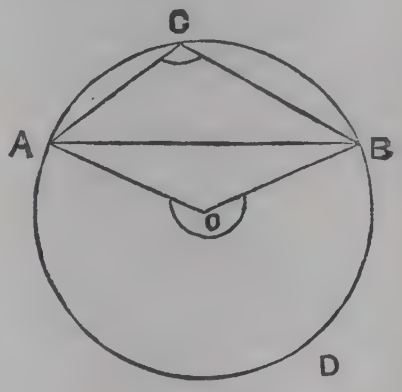


FIG. 54.

Again because the angle in the segment ACB and that in the segment ADB are together equal to 2 rt. angles, and because the angle in the segment ACB is less than a right angle, the angle in the segment ADB must be greater than a right angle. Hence we see that **(1) an angle in a segment greater than a semi-circle is less th n a right angle and (2) an angle in a segment less than a semi-circle is greater than a right angle.**

An independent proof of proposition (2) can be given as indicated in fig. 54.

Exercise VII (g).

1. What is the converse of the proposition that the angle in a semi-circle is a rt. angle? Prove it.

2. A number of right-angled triangles AP_1B, AP_2B, AP_3B... stand on the same hypotenuse AB. What is the locus of the points P_1, P_2, P_3..... ?

3. ABCD is a square. Show that the circle described on AC as diameter passes through the angular points A, B, C, D.

4. Show that the circum-centre of an obtuse-angled triangle falls without the triangle.

5. Show that the circum-centre of an acute-angled triangle falls within the triangle.

6. Show that the circum-centre of a right-angled triangle falls on a side of the triangle.

§ **54.** You have already seen that a chord at right angles to a diameter is bisected by that diameter. In the figure is shown a diameter with a number of lines cutting it at right angles. As the cutting line (generally called a **secant**) moves parallel to itself away from the centre, the two points of section (P_1, Q_1), (P_2, Q_2) ..... get nearer and nearer to one another until they coincide. The last line at R does not cut the circle in two points but merely touches it at one point, for the two points have become coincident. The st. line is then called a **tangent.**

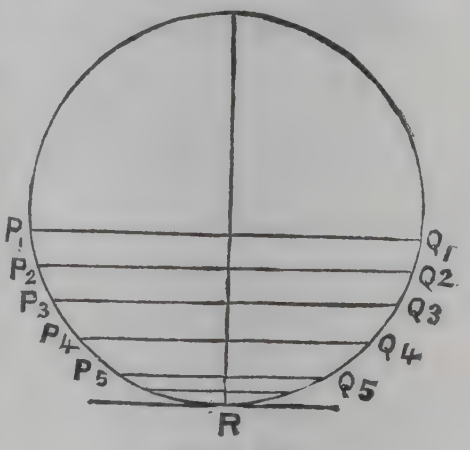


FIG. 55.

From the figure it is evident that **the tangent at a point to a circle is at right angles to the diameter passing through the point.**

Conversely ; **if a straight line is drawn at right angles to a tangent from its point of contact the line passes through the centre and is a diameter.**

In Fig. 55 suppose that the line AP revolves round A cutting the circle in P, Q, R... in successive positions. In the position AT it does not cut the circle but the other point of intersection of the secant approaches A and coincides with it. AT is thus **a tangent,** which may therefore be defined as *the limiting position of a secant when its points of intersection with the circle move up close to one another and ultimately coincide* or a secant which intersects a circle in two consecutive points.

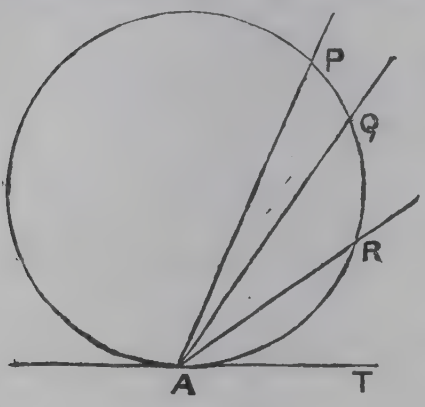


FIG. 56.

§ **55. To draw a tangent to a given circle from a given point**:—(1) when the point is on the circumference; (2) when the point is outside the circumference.

(1) Let A be the point on the circumference. Find O the centre. Join OA. From A draw AB at right angles to OA. Then AB is the tangent required.

(2) When the point A is outside the circle.

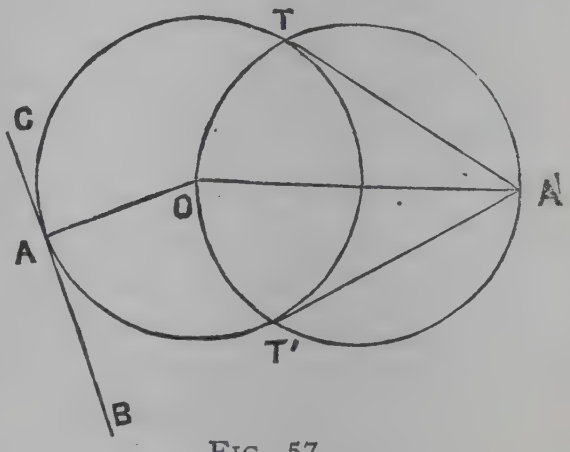


FIG. 57.

Analysis.—Suppose that AT is the tangent. Then OT the radius to T and AT the tangent must be at right angles.

In other words OTA is a right-angled triangle standing on the hypotenuse OA. ∴ the semi-circle described on

OA as diameter passes through T. Hence the following construction :—

Synthesis.—Join OA ; on OA as diameter describe a semi-circle cutting the given ⊙ in T. Join AT. Then AT shall be the required tangent.

If the circle on OA as diameter be completed, it will cut the given circle in another point, say, T′ ; then AT′ shall be another tangent from A.

Proof.—Join OT and OT′. The ∠OTA = 1 right angle being in a semi-circle, and the ∠OT′A = 1 right angle for a similar reason. ∵ TA and T′A are perpendicular to the radii to T and T′ ∴ they are tangents drawn to the circle.

Exercise VII (h).

1. *In the preceding figure show by comparing the two triangles OAT and OAT′ that the two tangents drawn from A to the circle are equal in length.*

2. *With the help of the same figure show that the centre of a circle which touches two intersecting straight lines is on the bisector of the angle between them.*

3. If l is the length of the tangent to a circle of radius r drawn from a point A whose distance from the centre is d, show that $l^2 = d^2 - r^2$.

4. Is it possible to draw a tangent to a circle from a point within a circle? Show that you can satisfactorily answer the question by a consideration of the relation $l^2 = d^2 - r^2$.

5. ABCD is a square. Inscribe a circle in it, *i.e.*, describe a circle in it so that each side of the square may be a tangent.

§ **56.** In a circle two chords AB and CD intersect at a point P whether inside or outside it. Find by measurement the products (the rectangles) AP . PB and CP . PD. Vary the position of the chords and compare AP . PB with

CP . PD in every position. You will find that AP . PB = CP . PD. The same may be proved thus :—

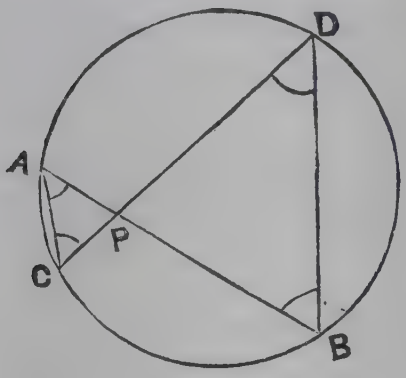


FIG. 58.

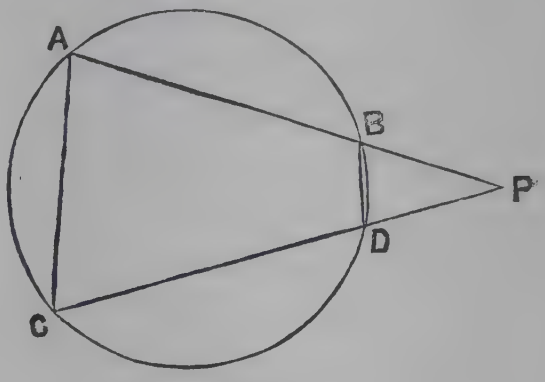


FIG. 59.

Join AC and BD.

In the △s ACP and BDP, ∠CAP = ∠PDB (being angles in the same segment in fig. 58; the exterior angle being equal to the opposite interior angle in fig. 59), and ∠APC = ∠BPD (being vertically opposite angles in fig. 58, and the common angle in fig. 59.)

∴ △s ACP and PBD are equiangular and similar.

$$\therefore \quad \frac{AP}{PC} = \frac{PD}{PB}.$$

$$\therefore \quad AP \,.\, PB = PC \,.\, PD.$$

Or thus : Let O be the centre.

Draw OK ⊥ to AB.

Then AK = KB

$$\begin{aligned} &AP \,.\, PB \\ &= (AK - PK)(KB + PK) \\ &= (AK - PK)(AK + PK) \\ &= AK^2 - PK^2 \\ &= (AK^2 + KO^2) - (PK^2 + KO^2) \\ &= AO^2 - PO^2. \end{aligned}$$

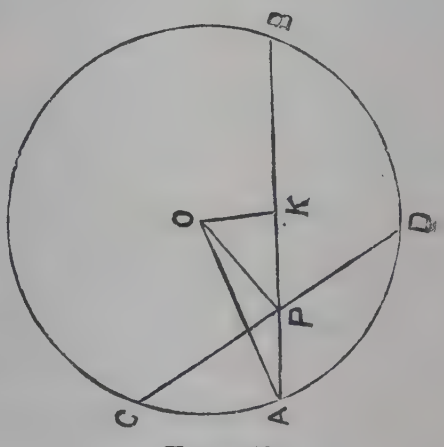


FIG. 60.

Similarly it may be shown that $CP \cdot PD = CO^2 - PO^2$. But $AO = CO$, $\therefore AO^2 - PO^2 = CO^2 - PO^2$.

$\therefore AP \cdot PB = CP \cdot PD$. (A similar proof for the case when P is outside the circle is left as an exercise to the student).

In fig. 59 suppose that the secant PBA so moves about P that B and A approach each other and ultimately coincide; so that the secant PBA becomes a tangent and the rectangle $PA \cdot PB$ becomes PA^2. Then the relation $PA \cdot PB = PD \cdot PC$ becomes $PA^2 = PD \cdot PC$ where PA is a tangent drawn to the circle from the point P, *i.e.*, **if from a point outside a circle a tangent and a secant are drawn, the square on the tangent is equal to the rectangle contained by the segments of the secant.**

The converse is also true, *viz.*:

If from a point outside a circle, two straight lines are drawn one cutting the circle and the other meeting it and if the rectangle contained by the segments of the secant is equal to the square on the line meeting it, the line that meets it shall touch it. (The proof is left as an exercise to the student.)

Cor. When AB a chord and CD a diameter at right angles to the chord intersect at P as in Fig. 61, $AP \cdot PB = CP \cdot PD$.

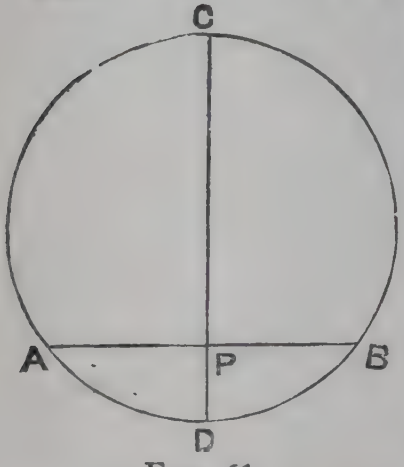


FIG. 61.

But AP = PB (because a chord at right angles to a diameter is bisected).

$\therefore AP^2 = CP \cdot PD$.

Or thus:—CAD is a semicircle. $\therefore$ the $\angle CAD$ is a right angle and AP is drawn perpendicular to the hypotenuse CD. And the square on the perpendicular is equal to the rectangle contained by the segments of the hypotenuse.

§ **57.** Let ABCD be a circle and EF a tangent to the circle at B and BD a chord dividing the circle into two segments. Measure the angle DBF and the angle in the alternate segment BAD. You will find that these are equal. Similarly measure the angle CBF and the angle in the alternate segment BDC. These will be found to be equal. Repeat the experiment with different circles drawing tangents and chords and measuring the corresponding angles. It will be found that **the angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.**

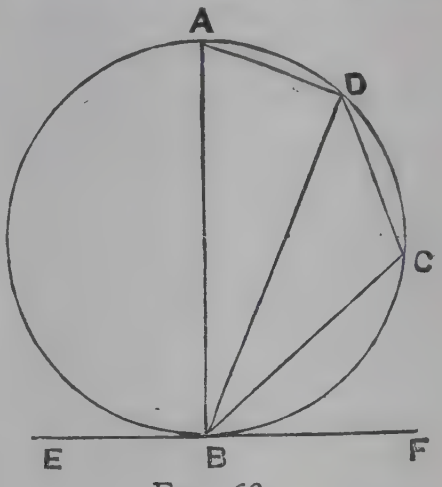


FIG. 62.

This may be proved by the following method of limits :

Let ACB be a segment of a circle of which AB is the chord ; and let PAT′ be any secant through A. Join PB. Then ∠BCA = ∠BPA being angles in the same segment, and this is true however near P may be to A.

If P moves up to and coincide with A, the secant PAT′ ultimately becomes the tangent AT and the angle BPA becomes the ∠BAT.

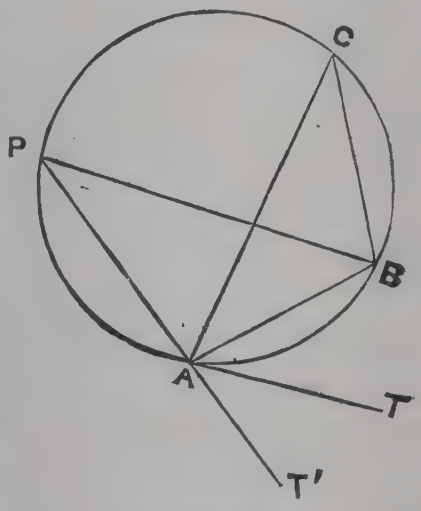


FIG. 63.

∴ ∠BAT = ∠BCA in the alternate segment.

Exercise VII (i).

1. In Fig. 60, if OP = d, OA = r, AP = a and PB = b, show that $ab = r^2 - d^2$.

2. If P be outside the circle, show that PA . PB = $d^2 - r^2$.

3. In Fig. 61, if PD = h, AP = c, show that $(2r - h)h = c^2$. Where r is the radius of the circle.

4. In the same figure, with the same notation, show that $2rh = c^2 + h^2$.

5. Show by applying the theorem of art. 56 that two tangents drawn to a circle from an external point are equal.

6. "If one side of a cyclic quadrilateral is produced the exterior angle is equal to the interior opposite angle." Deduce from this theorem the proposition proved in art. 57.

7. In Fig. (62), given that the $\angle FBD = 65^\circ$, find the values of the angles BAD, BCD and EBD.

8. Draw the tangent at a given point on a circle without first finding the centre of the circle.

9. ABC is a circle, A is a point on the circumference. From A draw a chord so that any of the segments formed may contain an angle of 85°.

§ 58. On a given straight line construct a segment of a circle containing an angle equal to a given angle.

Let AB be the given straight line, X the given angle. It is required to describe a segment of a circle containing an angle equal to X.

Analysis.—Suppose the segment APB is the required segment. At A draw a tangent AC. Then $\angle BAC$ = the angle in the alternate segment

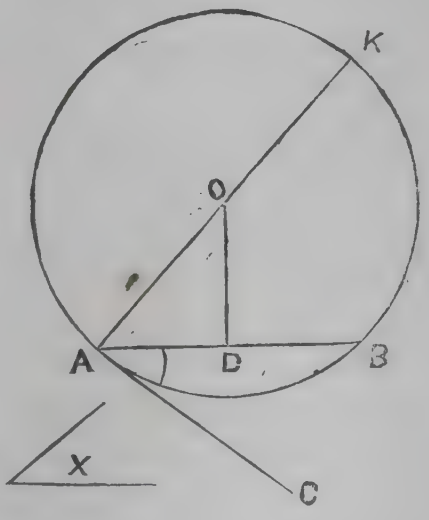


FIG. 64.

which must be equal to $\angle X$. Hence if we make $\angle BAC = X$, AC will be a tangent to the required segment.

Synthesis.—At A in BA make an angle $BAC = X$. ($\because$ AC is to be a tangent) the centre must be in a line AK drawn at right angles to AC from A. ($\because$ AB is a chord of the ⊙) the centre is in a line DO bisecting AB at right angles. $\therefore$ the centre is the point common to AK and DO, *i.e.*, O. With O as centre and OA or OB as radius describe a circle. Then AKB is the segment required.

Proof. $\because$ AC is at right angles to AK the diameter of the circle. $\therefore$ AC is a tangent.

$\therefore$ the angle in the segment $AKB = \angle BAC = \angle X$.

Exercise VII (j).

1. Without a protractor, construct a segment of a circle on a base $1\frac{1}{2}$ inches long containing an angle of 45°.

2. Without a protractor, on a given base 3·4 cm. long, construct an isosceles triangle with a given vertical angle, say, 30°.

3. On a given base 1·9 in. long, construct a △ having its vertical angle equal to 48° and one of its sides equal to 1·6 in.

4. On a given base 3·4 cm. long, construct a △ having its vertical angle equal to 110° and its altitude equal to 1·8 cm.

5. Construct a quadrilateral ABCD, given AB = 3·5 cm., BC = 4·7 cm., AC = 5·3 cm., $\angle D = 72°$, AD = 4·8 cm.

§ **59.** It may be easily shown by the principle of superposition that **in equal circles equal angles** at the centres or at the circumferences subtend **equal arcs**; and conversely; **equal arcs** are subtended by **equal angles** at the centres or at the circumferences.

Def. Equal circles are those which have equal radii and which have consequently their circumferences equal.

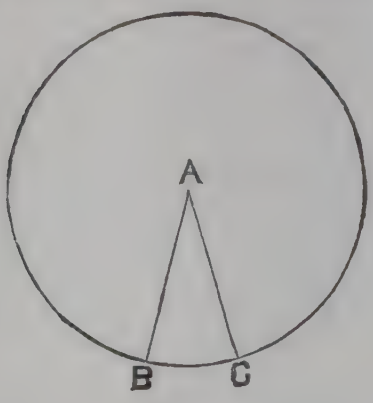


FIG. 65.

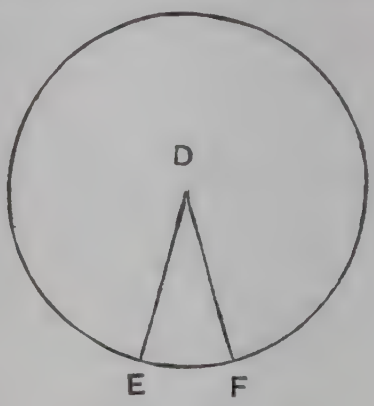


FIG. 66.

Let ABC and DEF be two equal circles with centres A and D and having the angles BAC and EDF equal. Then the arcs BC and EF shall be equal.

Apply the circle DEF to the circle BAC so that D may coincide with A and DE may fall on AB. Then since DE=AB, E coincides with B and since ∠BAC=∠EDF, AC falls on DF and F coincides with C. ∵ DF = AC; E coinciding with B and F with C, the arc EF must coincide with BC ∵ the circumferences must coincide being those of equal circles.

Conversely, if the arcs BC and EF be equal, then the ∠s BAC and EDF shall be equal.

Apply the circle EDF to the circle BAC so that D may fall on A and DE on AB and F and C may be on the same side of B.

∵ DE = AB; E will coincide with B. Again since the centres coincide and the radii are equal, the circumferences coincide. ∴ the arc EF falls on the arc BC and since they are equal F must coincide with C.

Thus ED and DF respectively coincide with BA and AC.

i.e. ∠EDF coincides with the ∠BAC.

∴ ∠EDF = ∠BAC.

∴ equal arcs are subtended by equal angles at the centre.

Since angles at the circumferences are halves of the angles at the centres on the same arcs, it may be inferred that equal angles at the circumferences, stand upon equal arcs, and conversely, equal arcs are subtended by equal angles at the circumferences.

It has been established in Part I (p. 388) that equal angles at the centre of a circle are subtended by equal chords; and conversely, that equal chords subtend equal angles at the centre. ∴ it may be inferred that **equal arcs are subtended by equal chords** and conversely **equal chords are subtended by equal arcs.**

§ 60. To bisect a given arc of a circle.

Let AB be the chord of the given arc of a circle. Bisect AB at right angles by CD meeting the arc at D. Then D shall be the middle point of the arc AB.

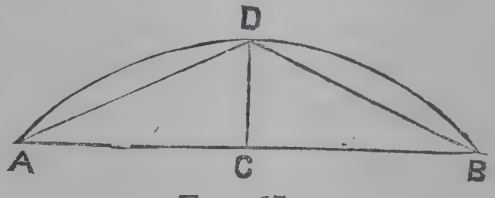


FIG. 67.

Join AD and BD.

Then ∵ AC=CB, CD is common and ∠ACD=∠BCD, each being a right angle.

△s ACD and BCD are congruent.

∴ chord AD = chord DB.

Equal chords are subtended by equal arcs.

∴ the arc AD = the arc DB,

i.e., the arc AB has been bisected.

Exercise VII (k).

1. Divide the circumference of a circle into (1) 3, (2) 4, (3) 5, (4) 6, (5) 8, (6) 10, (7) 12, (8) 15 equal parts.

2. Prove that an equilateral polygon inscribed in a circle must also be equiangular.

3. Prove that an equiangular polygon inscribed in a circle must also be equilateral.

4. Prove that the bisector of an angle in a segment passes through the middle point of the arc on which the angle stands.

5. AB and CD are parallel chords of a circle. Show that they intercept equal arcs and hence show that the figure ABDC is an isosceles trapezium.

Exercise VII (l).

1. Show that the only regular figures which may be fitted together so as to completely fill the space round a point are (1) equilateral triangles, (2) squares, (3) regular hexagons.

2. Three angles of a quadrilateral are $115\frac{1}{2}°$, $69\frac{1}{2}°$, $80°$. Find the fourth angle.

3. Two angles of a cyclic quadrilateral are $125°$ and $75°$. Find the remaining angles of the quadrilateral.

4. In a quadrilateral ABCD the angles A, C, D are respectively equal to 3B, 4B and 2B. Find all the angles.

5. Four angles of an irregular pentagon arc $30°$, $75°$, $128°$, and $136°$. Find the fifth angle.

6. Express, in degrees, the angles of an isosceles triangle in which each base angle is double of the vertical angle.

7. ABC is an equilateral triangle inscribed in a circle. D, E, F are points on the arcs opposite to A, B, C. Find the magnitude of each of the angles BDC, AEC and AFB.

8. In the preceding question, if ABC be any triangle inscribed in the circle, find the sum of the angles BDC, AEC and AFB.

9. A point O is such that its distances from the angular points A, B, C, D of a quadrilateral are 2·6 cm., 3·4 cm., 3·8 cm. and 4·4 cm. respectively. Draw the quadrilateral if $\angle AOB = 112°$, $\angle BOC = 85°$, $\angle COD = 105°$.

10. Repeat the same exercise with OA = 1·1″, OB = 1·2″, OC″=2·1″ and OD=2·5″, ∠AOB=25°, ∠BOC=40°, ∠COD=30°.

11. Construct a cyclic quadrilateral ABCD, given ∠ABC=90°, AC = 5 cm., AB = 3 cm., AD = 2 cm.

12. Construct a cyclic quadrilateral ABCD given ∠ABC=60°, ∠BAD = 80°, AB = 2·1″, AD = 2·4″.

13. Show that if a parallelogram be inscribed in a circle it cannot but be rectangular.

14. A point P moves so that a fixed straight line AB always subtends the same angle at P. Find the locus of P. Suppose that AB is 2 inches long and the angle APB = 60°. Of all the triangles AP_1B, AP_2B, AP_3B, etc., how many will be isosceles triangles with AB as the base? Find the magnitude of the angles of such an isosceles triangle. Hence construct the triangle.

15. Construct a segment of a circle containing an angle of 80° on a base 5·6 cm. long.

16. Construct a right-angled triangle having given (*a*) the hypotenuse = 5 cm., one side = 3·4 cm.

(*b*) the hypotenuse = 6 cm., the ⊥r on the hypotenuse from the right angle = 1·4 cm.

17. Construct a triangle given

(*a*) base = 3·4″, vert. ∠ = 30°, altitude = 2″.

(*b*) base=6 cm., vert. ∠=60°, median to the base=4·6 cm.

18. PQRS is a quadrilateral described about a circle, so that all its sides touch the circle. Show that PQ + RS = PS + QR. Measure the sides and verify.

19. If the sum of one pair of opposite sides of a quadrilateral is equal to the sum of the other pair, show that a circle can be inscribed within it.

20. Apply the test in question 19 to show that it is not possible to inscribe a circle in an oblong, but possible in a square.

21. Draw a tangent to a circle two inches and a half in radius, from a point distant three inches and a half from its centre.

22. Inscribe an equilateral triangle in a circle, and circumscribe a triangle to the circle by drawing tangents at the vertices of the inscribed equilateral triangle. Show that the circumscribed triangle is equilateral.

23. Describe a circle touching a line AB at C. How many circles can be described? Draw figures for the two cases when the circles touch (1) on the same side of the line; (2) on opposite sides of the line. What is the locus of the centres of all such circles? [Such circles are said to touch one another.]

24. A and B are points 2 inches apart on the circumference of a circle whose diameter is 4 inches. Draw tangents at A and B and measure the angle between the tangents. Can you give the magnitude of the angle without drawing the tangents?

25. A man stands on a plank resting on the top of a globe 24 ft. in radius. Supposing his eye to be 6 ft. from the surface of the globe, how far from his eye is the farthest part of the globe which he can see?

26. There is a circular wall whose diameter is 16 ft. A man stands at a distance of 9 ft. from the nearest point of the wall outside the room. How far from him is the farthest point of the wall which he can see?

27. There are two trees 15 ft. apart. The line joining the trees subtends an angle of 45° at a place P where a treasure is hidden. Are these observations sufficient for finding the hidden treasure? If not, suppose in addition you are given that the treasure is at a distance of 6 ft. from the line of the trees. Can you determine the spot where the treasure is? How many solutions will there be?

28. On a blackboard measuring 4 ft. by 3 ft. is drawn a triangle so that (1) one vertex is 18 inches from the top left-hand corner of the blackboard, and 24 in. from the bottom left corner; (2) another vertex is 24 in. from the top right-hand corner and 18 in. from the bottom right corner; (3) the third vertex is 24 in. from the bottom right corner and the angle at the vertex is a right angle. Draw a plan showing the position of the triangle relative to the sides of the blackboard. State your scale below the plan. What is the length of the hypotenuse?—(S. S. L. C. Exmn., 1911).

29. Draw a circle of radius 2·5 inches. Take a point 3 in. from the centre and draw a line from it to cut the circle so that the chord cut off may be 2·6 inches.

30. Circumscribe a square to a given circle.

31. As in exercise 30, circumscribe (1) a pentagon, (2) a hexagon, to a given circle.

32. An arch in the form of an arc of a circle is constructed across a stream 30 ft. wide. The highest point of the arch is 5 ft. above the chord of the arc. Calculate the diameter of the circle.

33. The radius of a circular arch is 50 ft. and its height (height above the chord of the arc) is 36 ft. Find the span of the arch. If the height is reduced by 16 ft., the radius remaining the same, by how much will the span be reduced? Check your result by a diagram taking 1″ to represent 20 ft.

34. If d denotes the shortest distance from an external point to a circle and t the length of the tangent from the same point, show that $d\ (d+2r)=t^2$. Hence find the diameter of the circle when $d = 1{\cdot}4''$ and $t = 2{\cdot}5''$ and verify your result by drawing a figure to scale,

35. If the horizon visible to an observer on a cliff 660 ft. above the sea level is $31\frac{1}{2}$ miles distant, find roughly the diameter of the earth. Hence find the approximate distance at which a bright light raised 90 ft. above the sea is visible at the sea level.

36. Find all the angles of Fig. 68, the magnitudes of the angles marked being given.

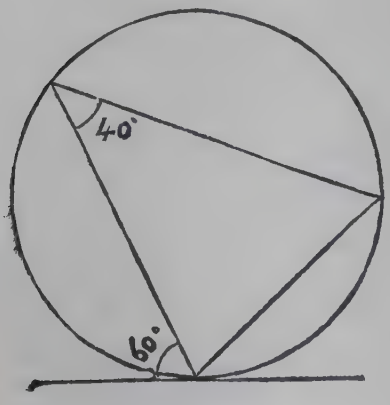


FIG. 68.

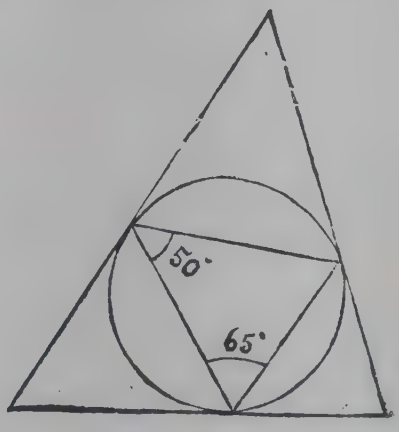


FIG. 69.

37. Find all the angles of Fig. 69.

38. In a given circle inscribe a triangle equiangular to a given triangle ABC.

39. P and Q are two points on the circumference of a circle. Through P and Q draw two equal and parallel chords.

40. Construct a $\triangle$ on a given base 4·8 cm. long, given the vertical angle to be 65° and also the length of the perpendicular from one of the extremities of the base to the opposite side to be 3·2 cm.

41. Construct a $\triangle$ on base 2·8 inches long, given the vertical angle $= 48^0$ and that the bisector of the vertical angle divides the base in the ratio of 3 : 4.

42. Inscribe an equilateral triangle in a circle. Bisect the three arcs. Join successively the extremities of the six arcs so obtained. Show that the figure formed is a regular hexagon.

43. Similarly get a regular decagon from an inscribed regular pentagon, and a regular octagon from an inscribed square.

44. Describe a circle of 4 cm. radius to touch two straight lines intersecting at an angle of 30^0. How many solutions are there ?

45. Describe a circle to touch two parallel straight lines distant 5·4 cm. from each other. What is the radius of the circle? How many solutions are there ?

46. What is the locus of the centres of all circles in Ex. 45?

47. What is the locus of the centres of circles touching two intersecting straight lines ?

48. Describe a circle so as to touch three given straight lines. How many solutions are possible ? If two of these are parallel, how many solutions are there ?

49. Inscribe a circle in a $\triangle$ whose sides are 6, 7, 8 cm. and measure the radius of the inscribed circle.

50. Describe the three *escribed* circles (a circle described so as to touch one side of a triangle and the other two sides produced is said to be escribed to the triangle) of the same triangle and measure their radii.

CHAPTER VIII.

SQUARE ROOT.

§ **61. Definition.** We have mentioned in the chapter on Multiplication in Part I that $a \times a = a^2$ and a^2 is called the square of a. Here a is said to be **the square root** of a^2. The square root of a number (N) is that number whose square is equal to the given number, *i.e.* N, *e.g.*, the square root of 64 is 8, because the square of 8 is equal to 64. The square root of N is symbolically written $\sqrt{N}$. Thus $\sqrt{36}$ means the square root of 36.

N.B. $\because$ $(+6)^2 = 36$ and $(-6)^2 = 36$. $\sqrt{36}$ is either $+6$ or -6.

Geometrically if the area of a square is taken to represent the given number, then the square root of the number will be represented by a side of the square.

The cube root of a number is that number whose cube is equal to the given number. Thus the cube root of 125 is 5 because $5^3 = 125$.

Exercise.—Oral.

Find the square root of

1. 49. **2.** 81. **3.** 121. **4.** 144. **5.** 169.
6. a^2b^2. **7.** $a^4b^4c^2$. **8.** a^6b^4. **9.** $x^2y^4z^2$. **10.** $x^{50}y^{64}$.

In this chapter a general method for finding square roots of large numbers will be given. The process by which the roots of numbers are found is known as **evolution** or extraction of roots.

If a table of numbers and their squares as shown below

Number	1	2	3	4	5	6	7	8	9	10	11	12	13
Square	1	4	9	16	25	36	49	64	81	100	121	144	169

be constructed it will be readily seen that all numbers in Arithmetic are not perfect squares. The first 5 square numbers are 1, 4, 9, 16 and 25; the numbers 2, 3, 5, 6, 7, 8 ..... are not square numbers. Thus we see that every number has a square but every number has not got an exact square root. The table also tells us that all square numbers end in 1, 4, 5, 6 or 9; hence it may be inferred that no number ending in 2, 3, 7 or 8 can have an exact square root.

We shall first proceed to find the square roots of numbers which are perfect squares.

One method of finding the square roots of exact square numbers is by factorisation.

Ex. Find the square root of 1936

$$1936 = 4\times4\times11\times11 = 4^2\times11^2$$

$\therefore$ the square root is 44.

Exercise VIII (a).

Find by factors the square root of

1. 324.	**2.** 361.	**3.** 576.
4. 1296.	**5.** 1764.	**6.** 2304.

§ 62. Number of digits in the square root of a number.

$$0^2 = 0$$

$$10^2 = 100$$

$$100^2 = 10{,}000$$

$$1000^2 = 1{,}000{,}000$$

$$10000^2 = 100{,}000{,}000.$$

Any number of one digit lies between zero and 10, and since $0^2 = 0$ and $10^2 = 100$, its square must lie between 0 and 100, *i.e.*, the square of a number of 1 digit con-

sists of 1 or 2 digits; similarly it may be seen that the square of a number of 2 digits consists of 3 or 4 digits and in general, the square of a number of n digits contains $2n$ or $2n-1$ digits. Conversely it is clear that the square root of a number between 0 and 100 should lie between 0 and 10, *i.e.*, the square root of a number of 1 *or* 2 *digits* contains 1 *digit*. Again the square root of a number between 100 and 10000 should lie between 10 and 100 $\therefore$ the square root of a number of 3 *or* 4 digits has 2 *digits*. Similarly the square root of a number between 10,000 and 1,000,000, should be between 100 and 1000, *i.e.*, the square root of a number of 5 or 6 *digits* has 3 *digits*. In general the square root of a number of $2n-1$ or 2 n digits has n digits.

If then in any given number whose square root is required, we mark off the digits into periods of two beginning from the units' place, the number of digits in the square root will be equal to the number of such periods. If the number of digits in a number is odd the last period will contain one digit only, *e.g.* the square root of the number $\dot{8}\ 76\ 38\ 49$ will consist of 4 digits.

Exercise.—Oral.

1. How many digits will there be in the square of
(*a*) 9. (*b*) 25. (*c*) 289. (*d*) 4402. (*e*) 10351.

2. How many digits will there be in the square root of
(*a*) 529. (*b*) 1024. (*c*) 11664. (*d*) 89415936

§ 63. General method of extracting the square root.—Suppose the square ABCD represents the number whose square root is to be extracted, then the square root will be represented by AB or BC; also suppose it is known that a^2 which is represented by BEGF is

very nearly equal to the number so that a is a first approximation to the square root of the number. The required square root BC is then evidently got by finding the additional amount EC (representing, say, b).

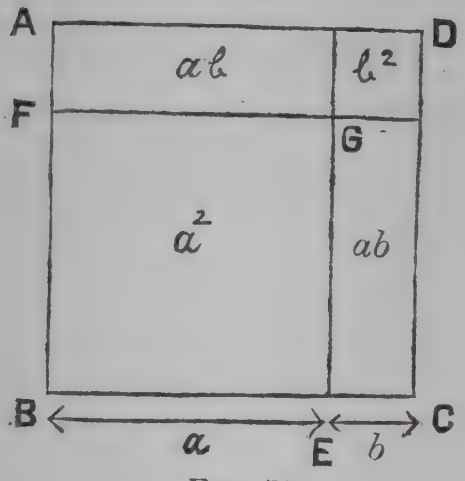


FIG. 70.

Fig. ABCD — fig. BEGF
= fig. AG + fig. GC + fig. GD
$= ab + ab + b^2$
$= 2ab + b^2$
$= 2ab$ very nearly

$$\therefore \quad b = \frac{\text{fig. ABCD} - \text{fig BEGF}}{2a}.$$

This gives the clue for the arithmetical method of extracting square roots.

Ex. Find the square root of 8836.

The square root evidently consists of 2 digits, because it gets divided into two periods.

		Square root.
		$90+4$
Fig. ABCD =	8836	8836 lies between 90^2 and 100^2.
Fig. BEGF =	8100	Write down the square of 90^2
Fig. ABCD—fig. BEGF =	736	
$2 \times 90 = 180$		$\frac{736}{180} = 4$, units digit or b.
Add 4		Multiply 184 by 4.
184	736	

Since there is no remainder $\sqrt{8836} = 94$.

Note. Fig. ABCD — fig. BEGF

$= (a+b)^2 - a^2 = 2ab + b^2$

i.e. $736 = b(2a+b) = 4\ (180 + 4)$

This is the reason why 4 is added to 180, *i.e.*, $2a$ and this sum is multiplied by 4.

Usually the work is shown thus—

$$\begin{array}{r|l} &)\dot{8}8\dot{3}6(94 \\ & 81 \\ \hline 184 & 736 \\ & 736 \\ \hline \end{array}$$

This method of extraction of square root is evidently based on a consideration of the identity $(a+b)^2 = a^2 + 2ab + b^2$. It may be exhibited in symbols thus :—

$$\begin{array}{r|l} & a^2 + 2ab + b^2 (a+b) \\ & a^2 \\ \hline 2a & 2ab + b^2 \\ \quad + b & \\ \hline 2a + b & b(2a+b) \text{ or } 2ab + b^2. \\ \hline \end{array}$$

We thus have the following rule for finding the square root :—

1. Divide the number into periods commencing from the right, *i.e.*, from the units' place.

2. Find the greatest number whose square does not exceed the first period of the number (from the left hand side). Set down this number in the margin on the right and its square under the period and subtract. Affix the next period. This whole excepting its last figure will be the trial dividend.

3. Multiply the digit of the square root already got by 2 and set down the result on the margin to the left and this will be the trial divisor.

4. Put down the quotient in the square root as well as after

the trial divisor. Multiply this whole written in the margin by the last digit of the square root.

Ex. Evaluate $\sqrt{549081}$.

Divide into periods as in the preceding example beginning from the right.

		549081	$\left[\begin{array}{l} 700+40+1 \\ \because\ 549081 \text{ lies between} \\ 700^2 \text{ and } 800^2 \end{array}\right.$
		490000	
		59081	$\frac{59081}{1400}$ suggest 4 for the tens-digit in the square root.
$2\times700=$	1400		
Add	40		
	1440	57600	Multiply 1440 by 40
		1481	$=549081-(740)^2$
$2\times740=$	1480		$\frac{1481}{1480}=1$ suggests 1 for the units digit.
Add	1		
	1481	1481	Multiply 1481 by 1.

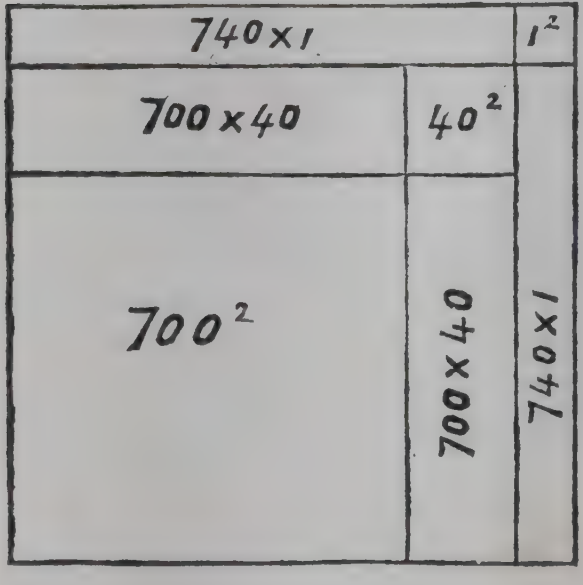


FIG. 71.

Since there is no remainder $\sqrt{549081}=741$.

The working is usually shown thus

$$\begin{array}{r|l} & \dot{5}\dot{4}\dot{9}0\dot{8}1(741 \\ & 49 \\ \hline 144 & 590 \\ & 576 \\ \hline 1481 & 1481 \\ & 1481. \end{array}$$

The process may be graphically illustrated as in figure (71) [not drawn to scale].

Note.—The above process may be extended to a number containing any number of periods.

§ **64. Square root of a decimal.**—There is no difference of procedure when finding the square root of a number partly integral and partly decimal as will be seen from the following example.

Extract the square root of 2704·416016.

$$2704{\cdot}416016 = \frac{2704416016}{1000000}$$

which shows that

$$\sqrt{2704{\cdot}416016} = \frac{\sqrt{2704416016}}{1000.}.$$

The square root of the number may be extracted ignoring the point; and because the denominator is 1000, there will be 3 decimal places in the answer.

Divide the number into periods beginning from the units' place as follows:

It is clear that in the square root there will be 2 digits in the integral portion and 3 in the decimal portion.

$$\begin{array}{r|l} & \dot{2}\dot{7}0\dot{4}\cdot4\dot{1}6\dot{0}1\dot{6} \mid 52004 \\ & 25 \\ \hline 102 & 204 \\ & 204 \\ \hline 104004 & 416016 \\ & 416016 \\ \hline & \end{array}$$

The required answer is 52·004.

Note.—104 does not divide into 41. Hence a zero is put in the square root and the next period brought down. Again 1040 does not divide into 416 ; hence another zero is put in the square root and the next period is brought down.

§ 65. Square roots of Vulgar fractions.—(1) If they are perfect squares reduce them to their lowest terms extract the square root of numerator and denominator separately *e.g.*, $\frac{25}{144}$

$$\sqrt{\frac{25}{144}} = \frac{\sqrt{25}}{\sqrt{144}} = \frac{5}{12}.$$

(2) Whether the fractions are perfect squares or not, multiply both the numerator and denominator by the denominator

Thus $$\sqrt{\frac{25}{144}} = \frac{\sqrt{25 \times 144}}{144 \times 144}$$

$$= \frac{5 \times 12}{144} = \frac{5}{12}.$$

(3) We may also change the vulgar fraction into a decimal and then extract the root.

Exercise.—Oral.

(*a*) How many decimal places will there be in the square of (1) 0·8 (2) ·0087 (3) ·00536 ?

(*b*) How many digits will there be in the integral and decimal portions of the square of (1) 3·6 (2) 9·008 (3) 760·0084 ?

(*c*) How many decimal places will there be in the square roots of (1) 0·64 (2) ·009801 (3) ·00001476 ?

(*d*) State the number of figures in the integral and decimal portions of the square roots of

(1) 57·76. (2) 1·3225. (3) 141·61.

Exercise VIII (b).

1. Find the sides of the squares the areas of which are (*a*) 2116 sq. ft. (*b*) 4225 sq. cm. (*c*) 9025 sq. yds.

2. What numbers when multiplied by themselves give as products (*a*) 8281 (*b*) 9409 (*c*) 11025 (*d*) 13225 (*e*) 13689 ?

3. Extract the square root of

(*a*) 19,321 (*b*) 692,224 (*c*) 5,774, 409 (*d*) 10,004,569 (*e*) 155,875,225 (*f*) 17,230,801 (*g*) 27,019,204 (*h*) 395,055,376 (*i*) 3,572,062,201 (*j*) 3,915,380,329.

4. Find the square roots of the following fractions :—(*a*) $\frac{841}{1024}$ (*b*) $\frac{2304}{5481}$ (*c*) $\frac{38642}{31752}$ (*d*) $410\frac{1}{16}$ (*e*) $5346\frac{55}{289}$.

5. Find the square roots of the following decimals :—

(*a*) ·1024 (*b*) 12·0409 (*c*) ·004225 (*d*) ·00099856 (*e*) ·24275329 (*f*) ·22071204 (*g*) 100·861849 (*h*) ·0012881 (*i*) ·0236144689.

§66. Square roots of numbers that are not perfect squares. The square roots of numbers which are not perfect squares can be obtained only approximately. And the method of extracting the roots of such numbers will become clear from the following examples :—

Ex. 1. Find, correct to two places of decimals, the square root of 2374.

First mark into periods 2374 ; the integral portion of the root will contain two digits. Since the root is required correct to 2 places of decimals we must work the root to 3 such places and the number of periods in the decimal portion of the given number must be 3. Thus the root is required of $2\dot{3}\dot{7}4\cdot0\dot{0}0\dot{0}0\dot{0}$.

$$
\begin{array}{rl|l}
 & 28\dot{7}40\dot{0}0\dot{0}0\dot{0}\,(48723 & \\
 & 16 & \\
\hline
88 & 774 & \\
 & 704 & \\
\hline
967 & 7000 & \\
 & 6769 & \\
\hline
9742 & 23100 & \\
 & 19484 & \\
\hline
97443 & 361600 & \\
 & 292329 & \\
\hline
 & 69271 &
\end{array}
$$

The square root, correct to 2 places of decimals is 48·72.

*§ 67. Approximations in square root.

Suppose a number, say 108, is very nearly an exact square; the first two figures in the square root are 1 and 0.

108 differs from 100 by 8, *i.e.*, by 8 %... ... ... (A)

$108 = 100 + 8 = (10)^2 + 2.\ 10.\ \frac{8}{2\,0} = (10 + \frac{8}{2\,0})^2$ very nearly.

$\therefore\ \sqrt{108} = 10 + \cdot 40.$

$\therefore\ \sqrt{108}$ differs from 10 by ·4, *i.e.*, by 4%... ... (B)

Comparing A and B it may be generalised that if a number N differs from another number n^2 by x % x being small $\sqrt{N}$ differs from $\sqrt{n^2}$ *i.e.*, from n by $\frac{x}{2}$ %.

Note. The principle involved may be made clear symbolically thus:—

We have seen that when m and n are integers $a^m \times a^n = a^{m+n}$. If this formula be assumed to be true when m and n are fractions, then $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$, *i.e.*, $a^{\frac{1}{2}}$ multiplied by itself gives a. $\therefore$ $a^{\frac{1}{2}}$ denotes the square root of a. Now if $N = n^2 + x$, then $\sqrt{N} = (n^2 + x)^{\frac{1}{2}} = \left[n^2 \left(1 + \frac{x}{n^2} \right) \right]^{\frac{1}{2}}$

$= n \left(1 + \frac{x}{n^2} \right)^{\frac{1}{2}} = n \left(1 + \frac{1}{2} \frac{x}{n^2} \right)$ or $n + \frac{1}{2} \frac{x}{n}$ nearly.

This result in addition to the principles already stated will be found to be the basis of a large number of approximate calculations.

Ex 1. Evaluate (*a*) $\sqrt{901}$. (*b*) $\sqrt{4949}$.

(*a*) $\sqrt{901} = \sqrt{900+1} = (30^2 + 1)^{\frac{1}{2}} = 30\left(1 + \frac{1}{30^2}\right)^{\frac{1}{2}}$

$= 30\left(1 + \frac{1}{2} \cdot \frac{1}{30^2}\right)$ nearly $= 30 + \frac{1}{2} \cdot \frac{1}{30} = 30 + \frac{1}{60}$.

(*b*) $\sqrt{4949}$

4949 differs from 4900 by 1%.

$\therefore$ $\sqrt{4949}$ differs from $\sqrt{4900}$ by $\frac{1}{2}$%.

= 70 increased by $\frac{1}{2}$%

= 70·35 approximately.

Ex. 2. $\sqrt{440} = \sqrt{441-1} = 21 - \frac{1}{42} = 21 - \cdot 024 = 20 \cdot 976$.

Ex. 3. On a railway incline the vertical rise is 1 ft. for every 200 ft. horizontal. What is the length of the incline for 400 ft. measured horizontally?

The length of the incline $= \sqrt{(400)^2+(2)^2}$

$= \sqrt{(400)^2+2 \cdot 400 \cdot \frac{4}{800}}$

$= 400 + \frac{4}{800} = 400 \cdot 005$ ft.

Exercise VIII (c).

Find the square root of—

1. $\frac{8}{5}$. **2.** $\frac{7}{12}$. **3.** $6\frac{2}{5}$. **4.** 3·256. **5.** 764·9.

2. If in example 3 worked above, the rise was 1 in. 360, what was the length of the incline if the rise was 6 ft.?

3. State the number of decimal places to which the following square roots can be found, the number under the root being approximate:

(*a*) $\sqrt{402 \cdot 8\ldots\ldots}$ (*b*) $\sqrt{902 \cdot \ldots\ldots}$ (*c*) $\sqrt{10000 \cdot 009}$.

4. Using a table of squares, evaluate:—

(1) $\sqrt{62}$. (2) $\sqrt{680}$. (3) $\sqrt{1025}$. (4) $\sqrt{1520}$.

(5) $\sqrt{2700}$. (6) $\sqrt{5470}$. (7) $\sqrt{13200}$. (8) $\sqrt{1400}$.

*§ **68.** If the square root of a number consists of $2n+1$ figures, when the first $n+1$ of these have been obtained by the ordinary method, the remaining n may be obtained by division only.

Let N denote the given number, a the part of the square root obtained by the ordinary method of root extraction consisting of $n+1$ figures (with n ciphers added), x the remaining part of the root (consisting of n figures). Then

$$\sqrt{N} = a + x$$
$$N = (a+x)^2 = a^2 + 2ax + x^2$$
$$\therefore\ N - a^2 = 2ax + x^2$$
$$\frac{N-a^2}{2a} = x + \frac{x^2}{2a}.$$

Now $N-a^2$ is the remainder after $n+1$ figures of the root (denoted by a) have been found and $2a$ is the trial divisor at that stage. It is clear that if the remainder is divided by the trial divisor at that stage, the integral part of the quotient gives x, the remaining part of the square root. $\left(\frac{x^2}{2a}\right.$ being a proper fraction.$\left.\right)$

The above demonstration implies that N is an integer with an exact square root; and exercises illustrative of the method could be easily given. But its extension to other cases is more useful and the following example is worked by the ordinary method and by the method explained in this article in order to illustrate such extension.

Ex. Find the first seven figures of $\sqrt{3}$.

$$
\begin{array}{rl|l}
 & \dot{3}\cdot\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0} & 1\cdot7320508 \\
 & 1 & \\
\hline
27 & 200 & \\
 & 189 & \\
\hline
343 & 1100 & \\
 & 1029 & \\
\hline
3462 & 7100 & \\
 & 6924 & \\
\hline
346405 & 176000 & \\
 & 173205 & \\
\hline
34641008 & 2795000 & \\
 & 2771280 & \\
\hline
\end{array}
$$

∴ the answer is 1·732051 correct to six decimal places.

According to the rule just now proved, 4 figures may be obtained by the ordinary method of extracting the square root and the remaining 3 by contracted division, thus :

$$
\begin{array}{rl|l}
 & \dot{3}\cdot\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0}\dot{0} & 1\cdot732 \\
 & 1 & \\
\hline
27 & 200 & \\
 & 189 & \\
\hline
343 & 1100 & \\
 & 1029 & \\
\hline
3462 & 7100 & \\
 & 6924 & \\
\hline
3\#\beta\# & 176 & 051 \\
 & 173 & \\
\hline
 & 3 & \\
 & 3 & \\
\hline
\end{array}
$$

∴ the answer = 1·732051.

Exercise VIII (d).

Find by the method of the above article the square root of : (1) 2. (2) 5. (3) 7. (4) 11. (5) 23 to seven figures in each case.

*§ **69. Evaluation of surds.** If the root of a number cannot be exactly obtained the root is called a *surd* or an irrational number.

Thus $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ are surds and are *incommensurable*, since the fractional part of such quantities cannot be exactly expressed as a fraction of a unit. The following formulæ will be found useful in the simplification of surds :—

1. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.
2. $\sqrt{a^2 b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$.
3. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.

Ex. 1. Evaluate $\frac{1}{\sqrt{5}}$ correct to 6 figures. If we substitute approximate numerical values for the surd $\sqrt{5}$, division should be carried on to a number of places. This involves serious labour.

The following is an easier plan :—

$\frac{1}{\sqrt{5}} = \frac{1 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{\sqrt{5}}{5}$ and $\sqrt{5} = 2{\cdot}2360680$ from tables or otherwise.

$$\therefore \frac{1}{\sqrt{5}} = \frac{2{\cdot}2360680}{5}$$
$$= {\cdot}4472136$$
$$= {\cdot}447214 \text{ correct to the sixth figure.}$$

Ex. 2. Evaluate $\frac{1}{3 - \sqrt{5}}$ to five places of decimals.

$$\frac{1}{3-\sqrt{5}} = \frac{1 \times (3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{3+\sqrt{5}}{9-5}$$
$$= \frac{3+\sqrt{5}}{4} = \frac{3+2{\cdot}2360680}{4}$$
$$= \frac{5{\cdot}2360680}{4} = 1{\cdot}3090170$$
$$= 1{\cdot}30902 \text{ correct to the fifth decimal place.}$$

Ex. 3. Evaluate $\frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{2}}$ to 4 places of decimals.

$$\frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{2}} = \frac{(\sqrt{5}+\sqrt{7})(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})}$$

$$= \frac{5+\sqrt{35}-\sqrt{10}-\sqrt{14}}{5-2}$$

$$= \frac{5+5{\cdot}9160728-3{\cdot}1622777-3{\cdot}7416574}{3}$$

$$= \frac{4{\cdot}0121377}{3} = 1{\cdot}3373792$$

$$= 1{\cdot}3374 \text{ correct to the 4th decimal place.}$$

Exercise VIII (e).

Evaluate—

1. $\sqrt{16}+\sqrt{17}$ to 5 places of decimals.
2. $\sqrt{27}+\sqrt{192}+\sqrt{12}-\sqrt{75}-\sqrt{108}$ to 5 places of decimals.
3. $\frac{\sqrt{15}}{\sqrt{20}}$ to 6 places of decimals.
4. $\frac{\sqrt{80}+\sqrt{180}}{\sqrt{8}+3\sqrt{2}}$ to 4 places of decimals.
5. $\frac{3}{\sqrt{19}-4}$ to 4 places of decimals.
6. $\frac{\sqrt{2}}{\sqrt{2}+1}$ to 6 places.
7. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ to 6 places.
8. $\frac{\sqrt{7}+\sqrt{2}}{\sqrt{7}-\sqrt{2}} - \frac{\sqrt{7}-\sqrt{2}}{\sqrt{7}+\sqrt{2}}$ to 6 places.
9. $\frac{2\sqrt{5}-3}{\sqrt{5}+4}$ to 6 places.
10. $\sqrt{7}+\frac{1}{\sqrt{7}}$ to 6 places.

§ 70. Graphic methods of finding the square root.

Method 1.

The square roots of all integers may be found in succession by the following construction :—

Draw OX = 1 inch and at O draw a line of unlimited length ⊥ to OX. Mark off $OA_1 = OX = 1$ inch. Then $A_1X^2 = 1^2 + 1^2 = 2$ square inches. $A_1X = \sqrt{2}$ inch.

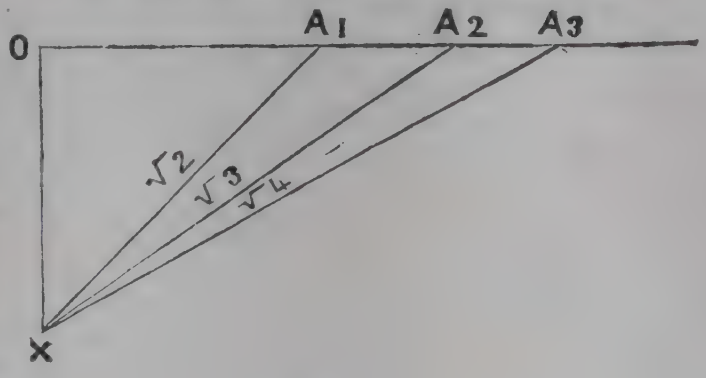


FIG. 72.

Mark off $OA_2 = A_1X = \sqrt{2}$.

Then $XA_2^2 = OX^2 + OA_2^2 = 1^2 + (\sqrt{2})^2 = 3$. $\therefore XA_2 = \sqrt{3}$.

Mark off $OA_3 = XA_2 = \sqrt{3}$, then $XA_3 = \sqrt{4}$ or 2. Proceeding similarly we may obtain $OA_1 = \sqrt{1}$

$OA_2 = \sqrt{2}$, $OA_3 = \sqrt{3}$, $OA_4 = \sqrt{4}$, $OA_5 = \sqrt{5}$ and so on.

Method 2.

We know that $\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2$

$$= \left(\frac{n+1}{2} + \frac{n-1}{2}\right)\left(\frac{n+1}{2} - \frac{n-1}{2}\right) = n = (\sqrt{n})^2.$$

$$\therefore \left(\frac{n+1}{2}\right)^2 = \left(\frac{n-1}{2}\right)^2 + (\sqrt{n})^2.$$

Hence if the hypotenuse and one side of a right-angled triangle be represented by $\frac{n+1}{2}$ and $\frac{n-1}{2}$, then the other side will represent $\sqrt{n}$. This property will enable us to find the square root of any arithmetical number.

Ex. Find graphically the square root of 7.

Here $n = 7$. $\frac{n-1}{2}$ and $\frac{n+1}{2}$ are $\frac{7-1}{2}$ and $\frac{7+1}{2}$, or 3 and 4. If, therefore, we

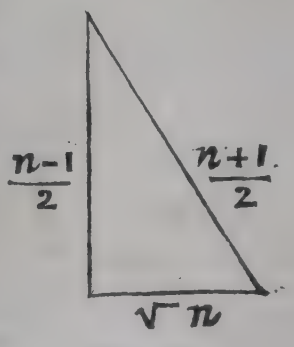


FIG. 73.

construct a right-angled triangle with 4 as the hypotenuse and 3 as one side, the other side will be $\sqrt{7}$.

Draw a straight line AB 4 cm. long. On it as diameter describe a semi-circle. With A as centre and AP = 3 cm., as radius describe a circle cutting the semi-circle at P. Join PB.

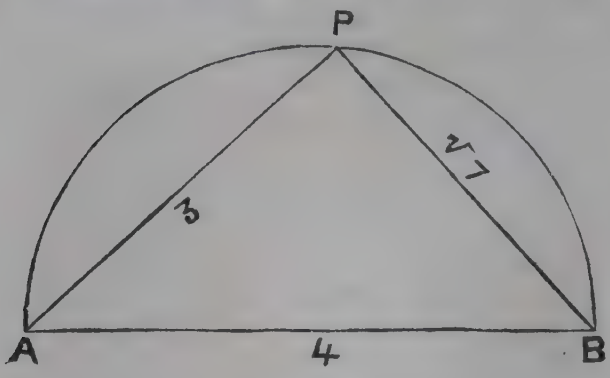


FIG. 74.

Then $PB^2 = AB^2 - AP^2 = 4^2 - 3^2 = 7$

$\therefore PB = \sqrt{7}$.

PB may be measured with the help of scale and will be found to be 2·6 cm.

Method 3.

Measure off a distance OA equal to the unit of length; produce AO to B so that OB represents the number N of which the root is required. Describe a semi-circle upon BA as diameter. Draw OC at right angles to AB to meet the circle at C. Then OC shall represent $\sqrt{N}$.

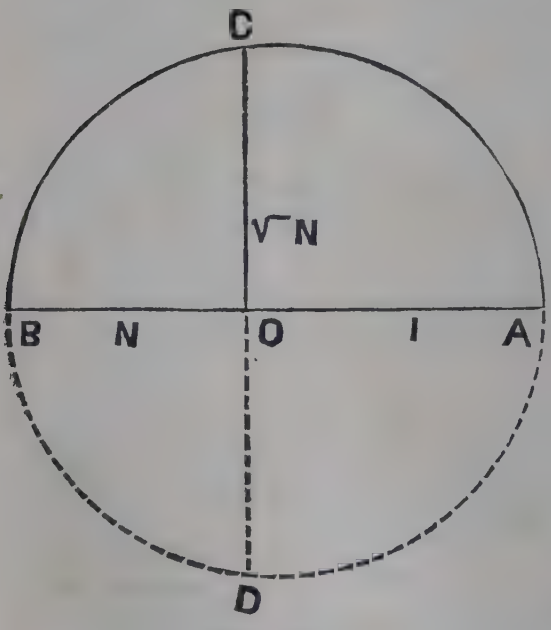


FIG. 75.

Proof. Produce CO to meet the circle at D. Then AB and CD are two straight lines intersecting at O.

$\therefore$ Rect. BO . OA = Rect. CO . OD.

But CO = OD (because it is a chord at right angles to a diameter and hence is bisected)

$\therefore$ BO . OA = CO . CO = CO^2

i.e. N . 1 = CO^2

$\therefore$ CO = $\sqrt{N}$

Or thus:

Measure off OA equal to the unit of length. Take a point B in OA or OA produced, so that OB represents the number N of which the root is required. Describe a semi-circle on OA or OB as diameter according as B falls within or without. Draw BC or AC at right angles to OB, according as B is in OA or OA produced, to meet the circle at C. Then OC shall represent $\sqrt{N}$.

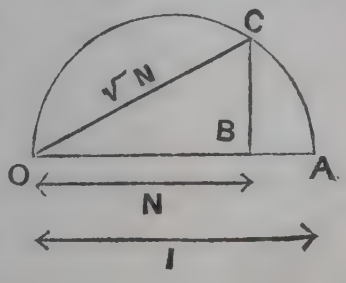


FIG. 76.

Proof. $\because$ CBA is a rt. angle

$\therefore$ the semi-circle described on CA as diameter passes through B, and OC at right angles to the diameter from its extremity C is a tangent to it.

Now OC is a tangent and OAB a secant drawn from O to the circle passing through A, C, B.

$\therefore$ $OC^2 = OA . OB = 1 . N$

$\therefore$ OC = $\sqrt{N}$.

Or thus:

$$OB . OA = OB . BA + OB^2$$
$$= BC^2 + OB^2$$
$$= OC^2, \text{ \&c.}$$

Exercise—Practical.

1. Find graphically the square root of 3, 5, 4, 16, 27 according to each of the above methods.

2. Verify the construction given in the last method when $N=6{\cdot}4$, $N=1{\cdot}96$, $N=3{\cdot}24$, $N=10$.

§ **71.** Many problems in Arithmetic and Algebra involve the extraction of square root as will be illustrated in the following examples :—

Ex. 1. A square field contains 5 acres. Find the distance from corner to corner.

Sol.—The distance from corner to corner is the diagonal.

(The diagonal)^2 = 2 × (the side)^2 = 2 × 5 acres = 10 acres).

∴ the length of the diagonal = $\sqrt{10 \times 4840 \text{ sq. yds.}}$ = 220 yds.

Ex. 2. A square field contains 40 acres. Find the length of a fence enclosing the field.

Sol.—Each side of the field = $\sqrt{40 \times 4840 \text{ sq. yds.}}$ = 440 yds.

Hence the length of the fence = 4 × 440 yds. = 1760 yds.

Ex. 3. There are two vertical poles which are 8 ft. apart. If the distance between their tops is 17 ft., find the difference between their heights.

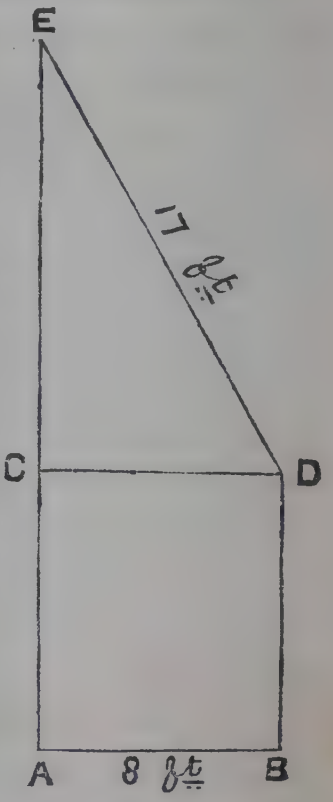


FIG. 77.

Sol.—Let AE and BD be the two poles and AB the distance between them. Draw CD parallel to AB.

AB is given to be 8 ft. and ED to be 17 ft.

We are required to find CE.

Since AB is 8 ft. in length, CD is also 8 ft. in length.

∴ EC = $\sqrt{ED^2 - CD^2} = \sqrt{17^2 - 8^2}$ feet = 15 ft.

Thus the difference between their heights is 15 ft.

Exercise VIII (f).

1 A square field contains 20 acres. Find the distance from corner to corner.

2. The product of 3 times a number and $4\frac{1}{5}$ times itself is $40937\frac{2}{5}$. Find the number.

3. In a school of 1156 boys there are as many classes as there are boys in each class. Find the number of classes in the school.

4. A vertical pole 15 ft. long casts a shadow of $66\frac{2}{3}$ ft. on the ground. Find the distance between the top of the pole and the extremity of the shadow.

5. Two vertical poles 8 ft. and 15 ft. high are 24 ft. apart. Find the distance between their tops.

6. Two trains start from T, the one going due North East and the other due South East at 60 and 11 miles an hour. When will they be $36\frac{3}{5}$ miles apart?

7. A ladder $32\frac{1}{2}$ ft. long will reach to a window 30 ft. from the ground on one side of the street; and if the foot of the ladder is retained in the same position it will reach to a window 26 ft. high on the other side. Find the breadth of the street and the distance between the two windows.

8. A house 47 ft. high is built upon a bank of a canal 35 ft. wide. How long must a ladder be which will reach from a point 5 ft. away from the edge of the canal, to a point in the wall of the house on the opposite bank which is 5 ft. from the top?

9. The top of a ladder which is $26\frac{9}{16}$ ft. long touches a certain point A $25\frac{1}{2}$ ft. from the ground. The ladder slips and is found to be 16 ft. farther from the bottom of the wall. Find the distance between A and the top of the ladder in the second position.

10. Find to within a foot, the length of a fence enclosing a square field of $7\frac{1}{2}$ acres.

11. Find the values of x which satisfy the following equations:

$$x + y = 7 \qquad xy = 12.$$

12. Prove that the square root of a proper fraction is greater than the fraction itself and its square is less than itself.

13. Take a piece of squared paper at least 8 inches each way ruled into tenths of an inch. Plot the curve $y = x^2$; taking 1 in.

for the unit on the x axis and 0·1 in. for the unit on the y axis, and plotting points on the curve corresponding to the values 0, 0·5, 1, 1·5, 2, 2·5.....8 of x. By means of the curve write down the square root of 6, 60, 7·4 and 74.

14. Use the curve to find the value of $\sqrt{750}$, $\sqrt{725}$.

15. Find to three significant figures, the square root of (1) 8, (2) 80, (3) 800, (4) 8000, (5) ·8, (6) ·08, (7) ·008, (8) $8^2 \times 10^6$, (9) 8×10^7, (10) 8×10^{-4}.

16. Archimedes stated that $\sqrt{3}$ is less than $\frac{1351}{780}$ and greater than $\frac{265}{153}$. Prove it.

17. Prove that $\sqrt{3} \times \sqrt{7} = \sqrt{21}$ by working out $\sqrt{3}$ and $\sqrt{7}$ to three significant figures, multiplying the result and by comparing it with $\sqrt{21}$ found by root extraction.

18. Two squares of sheet lead of sides 3 ft. and 3·5 ft. are melted to make a single square of the same thickness. Find to the nearest tenth of an inch the side of the resulting square.

19. A battalion of 3000 men is to be arranged in a solid square, how many men must stand out to make this arrangement possible? The arrangement would also be made possible by the addition of a certain number of men how many are required for this purpose?

20. Find to the nearest inch the edge of a cube whose total surface is (1) 18 sq. ft. (2) 24 sq. ft. (3) 32 sq. ft.

21. Show that $\sqrt{3} - 1$ is the square root of $4 - 2\sqrt{3}$ (by working out to three significant figures.

22. Find the difference between (*a*) $\sqrt{8} - 5$ and $\sqrt{8} - \sqrt{5}$; (*b*) between $\sqrt{2} + 5$ and $\sqrt{2} + \sqrt{5}$.

23. What would be the length of the side of a square representing an acre on an ordnance map (*a*) 3 inch to the mile (*b*) 5 inches to the mile?

24. Find the values of x which satisfy the following equations: —

$$(1)\ \frac{x}{639} = \frac{1028}{x}.$$

$$(2)\ \frac{1725}{x} = \frac{x}{86}.$$

25. The length of a rectangular court yard is 5 times its breadth. If its area is 3825 sq. yds, find the lengths of its sides to the nearest tenth of a yard.

26. ABC is a triangular field of which the base is 300 yds. and the height 99 yds. Find to the tenth of a yd, the side of a square field of equal area.

27. If a, b, c denote the sides of a triangle, its area may be found from the formula.

$$\text{Area of a triangle} = \tfrac{1}{4}\sqrt{(a+b+c)\,(b+c-a)\,(c+a-b)\,(a+b-c)}.$$

Find the area of a Δ whose sides are 30 ft., 28 ft. and 26 ft. The answer should be correct to the nearest sq. ft.

28. Give a graphical construction for finding (1) $\sqrt{18}$ (2) $\sqrt{10}$ (3) $\sqrt{21}$ and by means of your diagrams calculate these roots correct to three significant figures.

29. In measuring a square field I enter by accident in my note-book the length of one of the sides 1 pole too small. As a result the area found is 1 r. 39 poles too small. What is the true area?

30. ACDEF is a semi-circular portion of a terrace of which the diameter rafter AB is 6 ft. long and the cross rafters are Cc, Dd, Ee, Ff. Given that the distances of c, d, e, f from A are 1, 2, 4 and 5 ft., find the lengths of the cross rafters.

31. A is a point without a circle at a distance of 1·8 inches from the centre of a circle whose diameter is 3·2 inches. Calculate the length of the tangent from the point to the circle and verify your answer by measurement and drawing.

32. Find the length of (1) a chord of a circle whose diameter is 2·8 in., its distance from the centre being 5 inches; (2) the distance of a chord of 2 in. from the centre in the same circle.

33. Find by the contracted method the square root of 12484 correct to within a hundred thousandth; and thereby obtain $(111)^2$, $(11173)^2$ and the first seven figures of the square of 111·7318.

34. Show that $\sqrt{10}$ lies between $\frac{721}{228}$ and $\frac{4443}{1405}$ and that it differs from the former by a quantity greater than $\dfrac{1}{2\times228\times1405}$.

35. Assuming that the square of the speed of a body falling freely from rest is proportional to the height fallen through and that after falling through a height of 1 ft. the speed is 8 ft. per second, find, to within one hundredth of a foot per second, what the speed will be after falling through 920·5 ft.

CHAPTER IX.

COMPOUND INTEREST.

§ **72.** Interest may be calculated on a loan or a deposit in two ways. The interest on a sum of money for a certain time at a certain rate calculated as in Part I of this book, is known as **simple interest**; because there the interest is reckoned on the original principal only throughout the whole term of the loan so that the interest for the first year is the same as that for the second year or for any other year. In the other method of calculating interest called **compound**, the interest on the principal as it becomes due at the end of a year is added to the principal and the whole is conisdered due at the end of the year; so that in the next year interest will be charged on the amount due at the end of the first year and the amount due at the end of the second year is thus the sum of the amount due at the end of the first year and the interest thereon for the second year. Thus, in this method, the principal is continually being increased and consequently the interest.

The period at the end of which the interest is *compounded* with, or added on to, the principal, may be, instead of a year as above, any other stipulated period, say 6 months.

The examples worked below will illustrate the principles involved.

Ex. 1. What is the compound interest on Rs. 325 for 2 years at 5 $\%$ to the nearest pie? Find also the amount due at the end of the period.

Method 1.

Rs. 325 = Principal for the 1st year.
5
16·25 = Interest.....
325
341·25 = Amount at the end of 1st year or the Principal for the 2nd year.
5
17·0625 = Interest for the 2nd year.....
341·25
358·3125 or Rs. 358 5 as. = Amount at the end of the 2nd year.
Rs. 325 = Original Principal.
Rs. 33 5 as. = Compound Interest.

Ex. 2. Find the compound interest on Rs. 535 8 as. for $2\frac{1}{2}$ years at 6 %.

Rs. 535·5 = Principal for the 1st year.
6
32·130 = Interest.....
535·5
567·63 = Principal for the 2nd year.
6
34·0578 = Interest.....
567·63
601·6878 = Principal for the 3rd year.
3
18·0506 = Interest for 6 months.
601·6878
619·7384 = Amount at the end of $2\frac{1}{2}$ years.
535·5 Principal,
84·238 Interest correct to 3 places.
= Rs. 84 3 as. 10 p. Correct to the nearest pie.

Note 1. When the period of a loan contains a fraction of a year, it is usual to reckon only simple interest for the fractional part. Here, interest for 6 months alone is to be calculated in the third year and hence the multiplication by 3.

Note 2. The first three decimal places alone are sufficient to find annas and pies, hence the remaining places are discarded.

The following table enables one to see how the principal, interest and amount vary for each year and also according to what laws they increase or develop.

In the following table P denotes the principal, r the rate per cent.

Year.	Principal.	Interest.	Amount.
1st year.	P	$\frac{Pr}{100}$	$P+\frac{Pr}{100}$ or $P\left(1+\frac{r}{100}\right)$
2nd year.	$P\left(1+\frac{r}{100}\right)$	$P\left(1+\frac{r}{100}\right)\frac{r}{100}$	$P\left(1+\frac{r}{100}\right)+P\left(1+\frac{r}{100}\right)\frac{r}{100}$ $P\left(1+\frac{r}{100}\right)\left(1+\frac{r}{100}\right)$ or $P\left(1+\frac{r}{100}\right)^2$
3rd year.	$P\left(1+\frac{r}{100}\right)^2$	$P\left(1+\frac{r}{100}\right)^2\frac{r}{100}$	$P\left(1+\frac{r}{100}\right)^2+P\left(1+\frac{r}{100}\right)^2\frac{r}{100}$ $P\left(1+\frac{r}{100}\right)^3\left(1+\frac{r}{100}\right)=P\left(1+\frac{r}{100}\right)^3$
4th year.	$P\left(1+\frac{r}{100}\right)^3$	$P\left(1+\frac{r}{100}\right)^3\frac{r}{100}$	$P\left(1+\frac{r}{110}\right)^4$
5th year.	$P\left(1+\frac{r}{100}\right)^4$	$P\left(1+\frac{r}{100}\right)^4\frac{r}{100}$	$P\left(1+\frac{r}{100}\right)^5$
.....	.....	.....	.....
nth year.	$P\left(1+\frac{r}{100}\right)^{n-1}$	$P\left(1+\frac{r}{100}\right)^{n-1}\frac{r}{100}$	$P\left(1+\frac{r}{100}\right)^n$

Ex. 3. Find the compound interest on £500 for $2\frac{1}{2}$ years at 6 per cent. interest being payable half-yearly.

£500 = Principal for the 1st half year
3 [at 6 % rate for six months = at 3 % for a year]
15·00 = Interest for the 1st half year
500
515 = Principal for the 2nd half year
3
15·45 = Interest.....
515
530·45 = Principal for the 3rd half year
3
15·9135 = Interest.....
530·45
546·3635 = Principal.....4th half year
3
16·3909 = Interest.....4th half year
546·3635 (only 4 decimal places are retained)
562·7544 = Principal.....5th half year
3
16·8826 = Interest.....5th half year
562·7544
579·6370 = Amount at the end of 5th half year
500 = Original Principal
£79·637 = Compound interest

£79 12s. 9d. *Answer.*

**Or thus :*

This is the same thing as finding the compound interest for 5 half years at 3 per cent. per half year.

∴ by the formula given above, *viz.*

$$A = P\left(1 + \frac{r}{100}\right)^n = 500\left(1 + \frac{r}{100}\right)^5$$

$= 500 \times (1{\cdot}03)^5$.

It is clear that we require the answer correct to 3 places of decimals to find the shillings and pence in the answer ; ∴ since $(1{\cdot}03)^5$ is to be multiplied by 500, $(1{\cdot}03)^5$ should be found correct to 6 places.

It can be shown by continued multiplication that

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

$$\therefore \quad (1 + \cdot 03)^5 = 1 + 5 \times \cdot 03 + 10 \times (\cdot 03)^2 + 10 (\cdot 03)^3 + 5 (\cdot 03)^4 + (\cdot 03)^5$$

$$= 1 + \cdot 15 + 10 \times \cdot 0009 + 10 \times \cdot 000027 + 5 \times \cdot 0000008 \text{ [retaining only 6 places]}$$

$$= 1 \cdot 159274$$

$\therefore$ the amount $= 1 \cdot 159274 \times$ Rs. $500 =$ Rs. $579 \cdot 637$.

The amount is the same as given by the 1st method. In this question 6 $\%$ per annum is called the *nominal* rate. If interest is payable half yearly, the rate per cent. per half year is 3 $\%$ and £1 becomes $(1 \cdot 03)^2$ at the end of a year, so that 6 $\%$ per annum nominal rate, the interest being payable half yearly is the same as $6 \cdot 09$ $\%$ *effective* rate payable yearly.

Ex. 4. What sum will produce £126 2*s.* compound interest in 3 years at 5 per cent.

Int. = Amount — Principal = A — P

$$= P\left(1 + \frac{5}{100}\right)^3 - P = 126 \cdot 1 \text{ by the question}$$

i.e., $$P\left\{\left(\frac{21}{20}\right)^3 - 1\right\} = \frac{1261}{10}$$

$$P\left(\frac{21}{20} - 1\right)\left\{\left(\frac{21}{20}\right)^2 + \frac{21}{20} + 1\right\} = \frac{1261}{10}$$

$$P \cdot \frac{1}{20} \cdot \frac{441 + 420 + 400}{400} = \frac{1261}{10}$$

$$\frac{P}{20} \times \frac{1261}{400} = \frac{1261}{10}.$$

$$\therefore P = 800.$$

$\therefore$ £800 is the principal.

Another Method,

Let £1 be the principal ; it amounts to

$$£1\left(1+\frac{5}{100}\right)^3 = £\left(\frac{21}{20}\right)^3 \text{ or } £\frac{9261}{8000}.$$

∴ Compound interest on 1

$$= \frac{9261}{8000} - 1 = \frac{1261}{8000}.$$

$£\frac{1261}{8000}$ is the compound interest on £1.

∴ $£\frac{1261}{10}$ .....$£\frac{8000}{1261} \times \frac{1261}{10}$ or £800.

Ex. 5. Find, correct to a pie, the compound interest on Rs. 346 for 3 years at $4\frac{1}{2}$ per cent. per annum.

Sol.—Rs. 346 is the Principal

3	
10·38	the interest at 3 p. c. for 1 year.
5·19	,, ,, $1\frac{1}{2}$ p. c. ,,
346	
361·57	the amount at the end of 1 year.
3	
10·8471	the interest at 3 p. c. for the 2nd year.
5·4235	,, ,, $1\frac{1}{2}$ p. c. ,,
361·57	
377·840ø	the amount at the end of 2 yrs.
3	
11·3352	the interest at 3 p. c. for the 3rd year.
5·6676	,, ,, $1\frac{1}{2}$ p. c. ,,
377·8406	
394·8434	the amount at the end of 3 yrs.

∴ the interest at the end of 3 years is Rs. 48·843 or Rs. 48 13 as. 6 p. correct to a pie.

In the work we have retained only 4 places after the decimal, since the answer is required to be correct to 3 places only.

Ex. 6. Find, to the nearest integer, the sum of rupees which will amount to Rs. 12000 in 3 years at $3\frac{1}{2}$ p. c. per annum compound interest.

$$\text{Principal} \times \left(1 + \frac{3\frac{1}{2}}{100}\right)^3 = \text{Rs. } 12000.$$

$$\therefore \text{ Principal } = \frac{12 \times 10^3 \times 2^3}{(2{\cdot}07)^3}.$$

The integral portion of the answer will contain 5 significant figures. So, $(2{\cdot}07)^3$ must be expressed correct to six significant figures. The working is shown below

207	$12 \times 2^3 = 96$
207	$886974)\ 960000\ (\ 10823$
414	886974
1449	73026
42849	70958
207	2068
85698	1774
29994	294
886974	266

The answer is Rs. 10823.

Exercise IX (a).

Find the amount at compound interest (neglecting fractions of a penny or a pie)

1. Of £700 in 2 years at 5 per cent. per annum.
2. Of £3750 in 3 years at 6 per cent. per annum.
3. Of Rs. 4850 in 3 years at $4\frac{1}{2}$ per cent. per annum.
4. Of Rs. 2540 in 3 years at 4 per cent. per annum.

Find the compound interest (neglecting fractions of a penny or a pie)

5. On £1000 for 3 years at $4\frac{4}{5}$ per cent. per annum.
6. On Rs. 2080 for 3 years at $3\frac{1}{3}$ per cent. per annum.
7. On £433 6*s*. 8*d*. for 2 years at 5 per cent. per annum.
8. On Rs. 1250 for 3 years at $3\frac{1}{8}$ per cent. per annum.
9. On Rs. 50,000 for 3 years at $2\frac{3}{4}$ per cent. per annum.

Find the compound interest on,

10. £400 for 1 year at 5 per cent. per annum, interest being payable quarterly.

11. £9765 12*s.* 6*d.* for $1\frac{1}{2}$ years at 8 per cent. per annum, interest due half-yearly.

12. Rs. 640 for 9 months at 5 per cent. per annum, interest being payable quarterly.

13. Rs. 2500 for 2 years at 5 per cent. per annum, interest being payable half-yearly.

14. Find, to the nearest integer, the sum of Rupees that will amount to Rs. 3,000 in 3 years at $6\frac{1}{4}$ per cent. per annum compound interest.

15. What sum lent out at compound interest for 3 years at 5 per cent. per annum will amount to £926 2*s.* ?

16. What sum lent out at compound interest for 3 years at 4 per cent. will amount to £7030 8*s.* ?

17. On what sum will the compound interest for 3 years at 5 per cent. per annum be £78 16*s.* 3*d.* ?

18. On what sum will the compound interest for 4 years at 5 per cent. per annum be £862 0*s.* 6*d.* ?

19. Find the rate of interest at which Rs. 400 put out at comp. int. amounts to Rs. 441 in 2 years.

20. Find the rate of interest if £5468-15*s.* put out at comp. int. amounts to £5915 in 2 years.

21. Find the difference in the amounts of £150 for 4 years at 4 % per annum at (i) simple, (ii) compound interest.

22. If A borrows £200 from B at the beginning of each of 3 successive years, what sum will he have to return to him at the end of $3\frac{3}{4}$ years, comp. int. being reckoned at 5 % per annum for the first three years, and 4 % for the 4th year ?

23. A and B each lend £400 for 2 years at 4 % per annum comp. int. ; A's interest is payable half-yearly and B's yearly. How much does A receive more than B ?

24. I borrow £250 at 4 $^0/_0$ per annum comp. int., which I forthwith lend to a person at $4\frac{1}{2}$ % per annum comp. int.,

interest being payable half-yearly in the former case. If each loan continues for 2 years, what do I gain?

25. Find the difference between the amounts of £80 for 1 year at 4 % per annum comp. int. when the interest is payable (i) half-yearly, (ii) quarterly.

*HARDER EXERCISES ON INTEREST.

Ex. 1. A person borrows at the same time £250 and £350 at 6 and 5 per cent. per annum simple interest respectively on condition that the whole loan and interest be paid when the sum of the amounts of the two sums shall be £730. How long is the loan to continue?

Sol.—The interest on £250 for 1 year at 6 per cent.

$= \frac{£250 \times 6}{100} = £15.$

And the interest on £350 for 1 year at 5 per cent. $= \frac{£350 \times 5}{100}$

$= £17\frac{1}{2}$.

$\therefore$ the interest on the two sums for 1 yr. $= £32\frac{1}{2}$.

But the total interest is £730—£(250 + 350) or £130.

$\therefore$ the period required $= \frac{130 \times 2}{65} = 4$ years.

Ex. 2. A certain sum put out at compound interest amounts to £562 10*s.* in 1 year and to £656 2*s.* in 3 years. Find the sum lent and the rate of interest.

Sol.—By formula, amount at the end of 3 years.

$$= \left(1 + \frac{r}{100}\right)^3 \times P = \left(1 + \frac{r}{100}\right)^2 \times \left(1 \times \frac{r}{100}\right) \times P$$

$$= \left(1 \times \frac{r}{100}\right)^2 \times \text{(amount at the end of 1 year)}.$$

It will be clear from this that £562 10*s.* amounts to £656 2*s.* in 2 years at comp. int.

$$\therefore \left(1 + \frac{r}{100}\right)^2 = \frac{656{\cdot}1}{562{\cdot}5} = \frac{729}{625} = \left(\frac{27}{25}\right)^2 = \left(1 + \frac{8}{100}\right)^2$$

Hence $r = 8$ or the rate is 8 p. c.

Since the amount at the end of 1 year is £562 10*s*., and the rate is 8 p.c., the principal may be found to be £520 16*s*. 8*d*.

Ex. 3. In how many years will £50000 amount to £56243 4*s*. at 4 per cent. per annum comp. int.

If n be the required number of years,

then $£56243{\cdot}2 = (1 + {}_{100}^{4})^n \times £50000$

But $\frac{56243{\cdot}2}{50000}$ or 1·124864

and $\left(1 + \frac{4}{100}\right)^n = (1{\cdot}04)^n$.

Here we should ascertain what power of 1·04 is 1·124864. We have only to divide 1·124864 by 1·04, the quotient by 1·04 and so on, till we get unity for the last quotient; and the number of divisions will show the number of years. Thus the number of years is found to be 3. [*Vide N.B.* below.]

Another Method.

£50000	= the principal for the 1st year.
4	
2000·øø	= Interest.....
50000	
52000	= Principal for the 2nd year.
4	
2080·øø	= Interest.....
52000	
54080	= Principal for the 3rd year.
4	
2163·2ø	= Interest.....
54080	
56243·2	= Amount at the end of the 3rd year.

Hence it amounts to £56243 4*s*. in 3 years.

N.B. Here the first method appears to be easier because the period is an exact number of years. But if the period contains any fraction of a year, the second method is preferable—suppose the time in any one case is $3\frac{1}{2}$ years. Then the amount will certainly be between that for 3 years and that for 4 years. The excess of the given amount over that for 3 years can be ascertained by the second method, and the interest for 1 year on the amount for 3 years may be determined. By proportion the number of months can be obtained.

Ex. 4. A certain manufacturer owns machinery to the value of £2520. Every year, the machinery depreciates in value by 5 %. What will be the value of his machinery at the end of three years?

£2520 = Value of machinery at the beginning.

£126·00 = Depreciation in the 1st year.

£2394 = Value at the beginning of the 2nd year.

£119·70 = Depreciation in the 2nd year.

£2274·3 = Value at the beginning of the 3rd year.

£113·715 = Depreciation in the 3rd year.

£2160·585 = Value at the beginning of the 4th year or at the end of the third year.

= £2160·6 approximately.

Ex. 5. At what rate per cent. per annum compound interest will £7000 amount to £7571 4*s.* in 2 years?

$$\text{Now } £7571{\cdot}2 = \left(1 + \frac{r}{100}\right)^2 \times £7000$$

$$\therefore \left(1 + \frac{r}{100}\right)^2 = \frac{7571{\cdot}2}{7000} = 1{\cdot}0816$$

$$\therefore \left(1 + \frac{r}{100}\right) = \sqrt{1{\cdot}0816} = 1{\cdot}04.$$

$\therefore$ Rate per cent. = 4.

N. B.—It is not proposed to give examples requiring extraction of other roots than the square root, inasmuch

as the student will, in the later stage of study, learn the general method by the use of logarithms.

Ex. 6. The following details taken from "Interest tables" give approximately the amount of Re. 1 at 5 per cent. per annum for different periods.

Number of years.	0	2	4	6	8	10	15	20	25
Amount at simple interest.	1	1·1	1·2	1·3	1·4	1·5	1·75	2	2·25
Amount at compound interest.	1	1·10	1·22	1·34	1·48	1·63	2·07	2·65	3·37

Illustrate these data graphically and use the graphs to find the amount of £100 in 12 years (i) at simple (ii) at compound interest. Also find from the graphs in how many years a sum of money will double itself (i) at simple (ii) at compound interest.

In Fig. (78) the years are marked on the horizontal axis, 0·2 of an inch being taken to represent 1 year. On the vertical axis 1 inch is taken to represent ·25 of a Rupee. Since when $n = 0$, $A = 1$, in each case both graphs pass through the point (0, 1) and this point may be conveniently taken as the origin. The points may be plotted and the graphs drawn as shown in the figure. OS is the graph of the amounts at simple interest and OC that of the amounts at compound interest.

When $n = 12$, from the simple interest graph, $A = 1{\cdot}6$ and from the compound interest graph $A = 1{\cdot}75$.

Again when $A = 2$, $n = 15$ in the compound interest graph, and in the simple interest graph $A = 2$ when $n = 20$.....

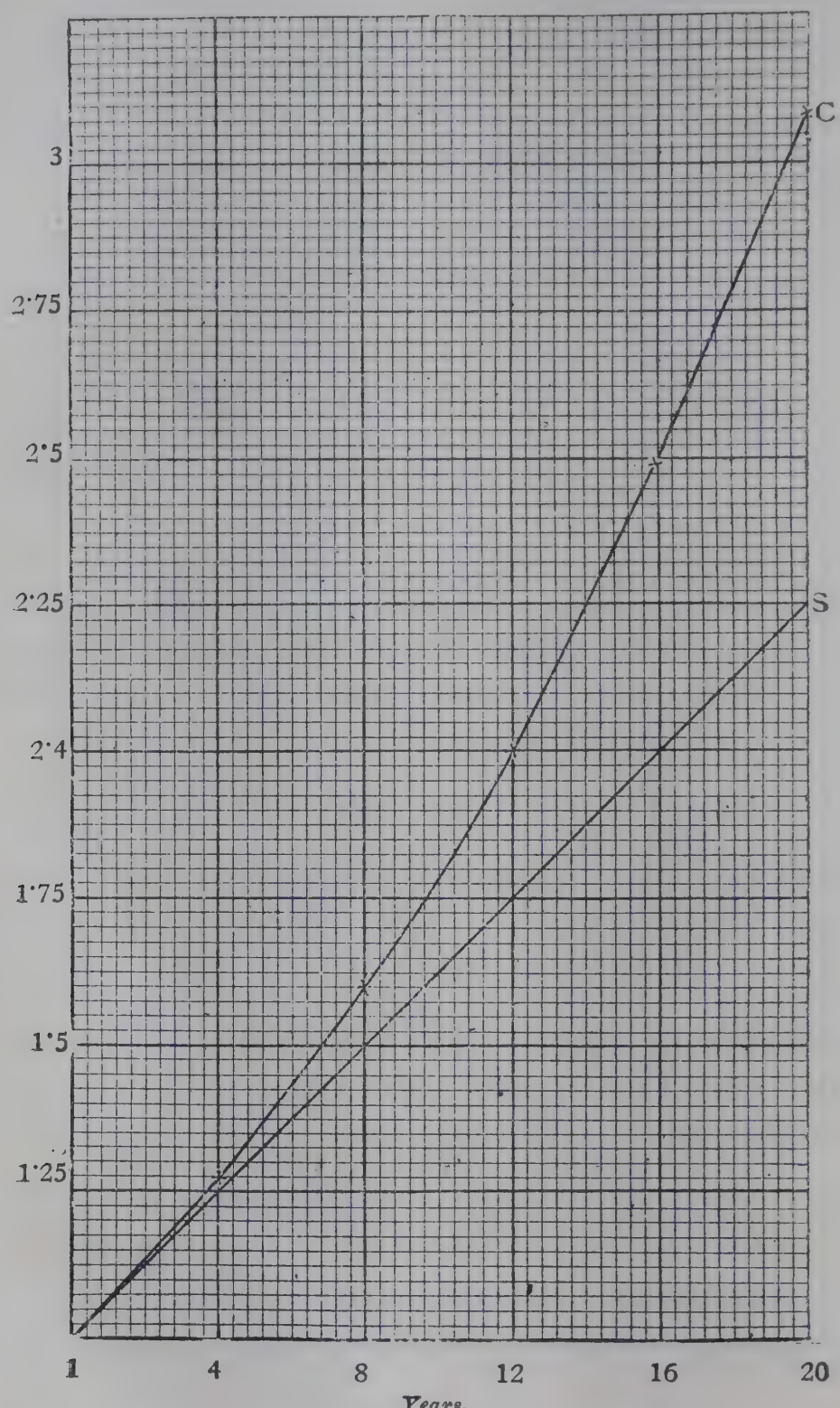


FIG. 78.

Note.—The simple interest graph being a straight line can be produced indefinitely beyond S and thus can be used to read off amounts for any number of years. In the case of compound interest graph, the curve is not known beyond the portion indicated in the figure and is therefore unsuitable for extrapolation.

Exercise IX (b).

1. Find the effective rate when the nominal rate per annum is as follows :—

(*a*) 6 % payable quarterly.

(*b*) 5 % yayable half yearly.

2. In what time will £64 amount to £81 at comp. int., the rate of interest being 12½ per cent. ?

3. In how many years will £27 amount to £64 at 33·3 % comp. int. ?

4. At 4 % for 3½ years, prove that the amount (A) = P × $(1{\cdot}04)^3 \times 1{\cdot}020$. Hence find the present value to the nearest rupee of Rs. 18050 due 3½ years hence at 4 per cent.

5. The census is taken every ten years, and it is found that at each census the population of a certain town is greater by 12 p. c. than the population at the preceding census. If it was 1,842,645 in 1891, what was it in 1911, 1881 ?

6. From the following table draw graphs to show the amounts at simple and compound interest of £1 at 4 % : —

Number of years.	0	5	10	15	20	25	30	35	40	45	50
Amount at simple interest.	1	1·20	1·40	1·60	1·80	2·00	2·20	2·40	2·60	2·80	3·00
Amount at compound interest.	1	1·21	1·48	1·80	2·19	2·66	3·24	3·94	·80	58·4	7·10

Find (i) the amount at compound interest of Rs.100 in 36 years ; (ii) in how many years the accumulated interest on Rs. 100 will amount to Rs. 300.

7. The yearly output of a gold mine decreases every year by 6 % of its amount during the previous year. Given that the first year's output is 2,500,000, find the output in the 4th year.

8. The population of a certain town is 1,500,000 in a certain year. Annually it loses 3 % by death, and gains 5 % by birth, and annually a fixed number 1500 emigrate. Find the population after the lapse of 3 years.

9. In a certain factory the machinery and plant were valued at Rs. 25000. If it is supposed to depreciate each year at 10 % of the value at the beginning of the year, find what its value would be after 4 years.

10. One vessel contains 10 gallons of wine and another 5 gallons of water. One gallon is taken from each and transferred to the other. Find the proportion of water to wine in the first vessel after the operation of transferring liquid from one vessel to the other has been performed three times in all.

11. ABCD is a square. The sides are bisected and the middle points joined to get another square $A_1B_1C_1D_1$; from $A_1B_1C_1D_1$ another square $A_2B_2C_2D_2$ is derived in the same manner. Find the area of $A_6B_6C_6D_6$ in terms of the area of the original square ABCD.

12. OR is the radius of a circle of which O is the centre. OR is bisected at R_1 and a second circle is described with OR_1 as radius, a third circle is derived in the same way from the 2nd circle and so on. Compare the area of the circle whose radius is OR_5 with that of the original circle (the area of a circle is proportional to the squareof the radius).

13. A ball bounces vertically and rises to heights which gradually decrease in such a way that the height in any one bounce is 60 % of the height in the preceding bounce. What is the height of the 6th bounce if the ball be first thrown up to a height of 5 ft.

14. If a national debt of £484,000,000 has a twelfth part of its then existing amount paid off each year, how soon will it be reduced to less than half its original amount?

15. A sum of Rs. 200 is deposited in a bank at the beginning of every year and compound interest is charged at 5 %. Find the amount at the end of 4 years.

16. A sum of money was put out at compound interest, the first year's interest was Rs. 32·13 and the third year's interest was 366·101268. Find the principal and the rate per cent.

17. A man borrows Rs. 2500 at 6 % per annum compound interest. He pays Rs. 500 at the end of each year. When will he clear his debt and what is his last instalment?

18. We have seen that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots$$

Hence find the value of

(*a*) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots\ldots$.

(*b*) $\cdot 9 + \cdot 09 + \cdot 009 + \cdot 0009 + \cdot 00009 \ldots\ldots\ldots$

CHAPTER X.

SIMULTANEOUS EQUATIONS AND PROBLEMS.

§ **73.** In part I, Chapter XIX you have been taught to plot a number of points whose co-ordinates are connected by an equation of the form $y = 2x + 1$, the values of y being obtained by giving different values to x and entered in a tabular form with the corresponding values of x. We shall now show that an equation of the form $y = 2x + 1$ always represents graphically a straight line; hence such equations are called *linear* equations. As before give to x some values, say, 4, 3, 2, 1, 0, — 1, —2, —3, then $y = 9, 7, 5, 3, 1, -1, -3, -5$. Arrange the results as follows :—

x	4	3	2	1	0	— 1	— 2	— 3
y	9	7	5	3	1	— 1	— 3	— 5

Plot the points (4, 9), (3, 7), (2, 5), (1, 3), (0, 1), (— 1, — 1) (— 2,— 3), (— 3, — 5). These points appear to be on a straight line AB. (Fig. 79). This may be verified by applying a ruler to the figure. Take any other solution, *i.e.*, pairs of values of x and y satisfying the equation $y = 2x + 1$, *viz.*, 10 and 21. Plot the point (10, 21) you will find that it lies on AB produced. Similarly the point (8, 17) corresponding to the solution $x = 8, y = 17$, of the equation lies on AB. Thus (1) *every point whose co-ordinates satisfy the equation*

$y = 2x + 1$ lies on the straight line AB if produced far enough both ways.

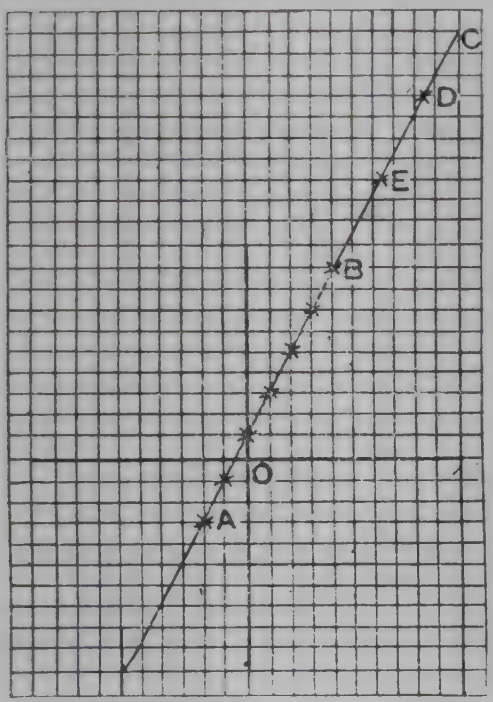


FIG. 79.

Again take any point in AB whose co-ordinates we can read off easily, for instance the point E (6, 13). You will find that these values of x and y satisfy the equation $y = 2x + 1$. *Thus* (2) *the co-ordinates of every point in the line AB satisfy the equation* $y = 2x + 1$.

From (1) and (2) it follows that if the co-ordinates of a point do not satisfy the equation the point does not lie on the straight line and that the co-ordinates of a point outside the line do not satisfy the equation. Thus the equation $y = 2x + 1$ is found to correspond to a certain straight line AB which is called the graph of the equation $y = 2x + 1$.

Similarly every equation of the first degree represents a straight line.*

How to draw the graph of a linear equation; say $2y = 3x + 8$.

Since the graph is a straight line, it is enough if two points are found on the graph. To get two pairs of values of x and y for determining the two points, we

* A formal proof of this is beyond the scope of this book.

put successively $x = 0$ and $x = 2$ in the equation, which give to y the values 4 and 7 respectively. $\therefore$ points (0, 4) and (2, 7) are points on the line. Plot these points and join them. (Fig. 80).

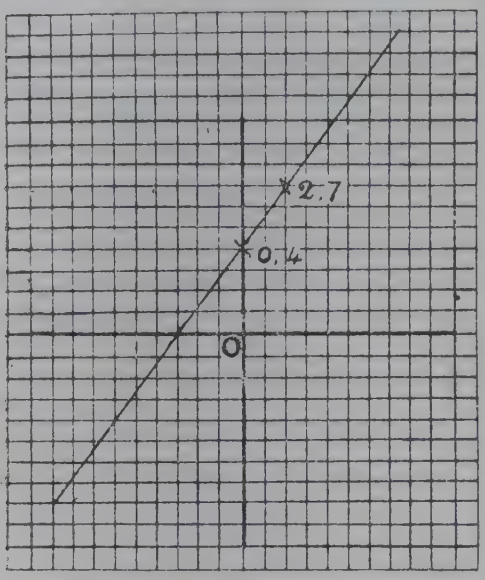


FIG. 80.

Or thus :

Put $x=0$. We have $y=4$ and put $y = 0$, we have $x = -2\frac{2}{3}$ *i.e.*, the point to (0, 4) on the y axis and $(-2\frac{2}{3}, 0)$ on the x axis are points on the straight line. $\therefore$ the join of these two points is the required graph.

§ 74. Graphical solution of simultaneous equations.

When only one equation is given connecting two quantities x and y, we have seen above that we have many solutions ; in fact corresponding to every point on the graph, there is a solution. Hence given only one equation the solution becomes indeterminate. But if we are given a second equation also, connecting x and y, we can definitely find values of x and y satisfying both the equations as shown below.

Solve graphically the equations

$$y - 2x = 1 \quad \ldots \quad \ldots (1)$$

$$y - x = 2 \quad \ldots \quad \ldots (2)$$

We draw the graphs of (1) and (2), which are straight lines. To draw the graph of (1) we find two points on it, *viz.* (0, 1) and (4, 9). Plot and join them. We get the line AB (Fig. 81) which is the graph of (1).

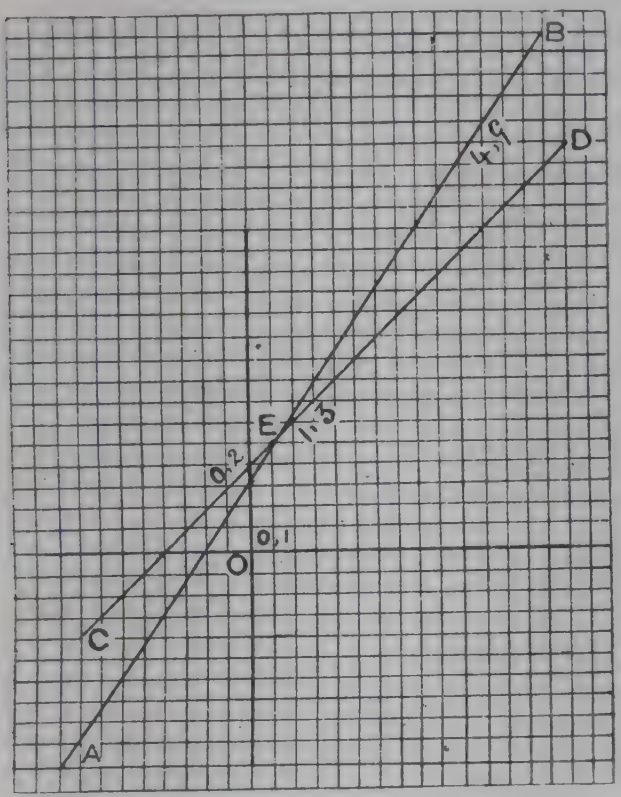


FIG. 81.

In (2) when $x=0$, $y = 2$ and when $x = 4$, $y = 6$, plot the points (0, 2), (4, 6) and join them; this is the graph CD (Fig. 81) of (2). The lines AB, CD intersect at E whose co-ordinates, namely, 1, 3 satisfy both (1) and (2) for E lies on both the lines. Hence (1, 3) is the solution of the given equations.

When two or more equations are satisfied by the same values of the unknown quantities they are called **simultaneous equations**.

Exercise X (a).

Solve graphically :—

1. $y = 3x - 1$
   $y = x + 3$.
2. $y - x = 0$
   $y + x = 0$.
3. $3x - y = 1$
   $2y + x = 5$.
4. $3x - 2y = 1$
   $4x + 3y = 24$.
5. $4x + 6y = 61$
   $7x = 9y - 25$.
6. $5x + 5y = 0$
   $2x + 5y = -29$.

§ 75. Algebraic solution of simultaneous equations.

One method of solving simultaneous equations is by '*Elimination*.' Elimination generally means getting rid of a quantity or quantities. Here we get rid of one unknown quantity and the result of this operation (usually known as the eliminant) gives us an equation containing only one unknown quantity. There are three methods of eliminating one of the unknown quantities—

(1) By equalising co-efficients and adding or subtracting.

(2) By substitution.

(3) By comparison.

Method (1).

Ex. 1. Solve

$$5x + 9y = 23 \quad (1)$$
$$7x + 6y = 19 \quad (2).$$

To eliminate x we multiply (1) by 7 and (2) by 5. This makes the coefficients of x in both the equations equal. This gives

$$35x + 63y = 161$$
$$35x + 30y = 95.$$

Subtracting $\quad 33y = 66$

$$y = 2.$$

To find x we may proceed similarly or substitute this value of y in either of the given equations, say in (1)

$$5x + 9 \times 2 = 23$$
$$\therefore \quad 5x = 5$$
$$x = 1$$
$$\therefore \quad x = 1 \; y = 2 \text{ is the required solution.}$$

Verification. $5x + 9y = 5.1 + 9.2 = 23$

$7x + 6y = 7.1 + 6.2 = 19$

The student is advised to draw the graphs of the two equations and thus verify the solution.

Ex. 2. $4y + 9x = -1$ ... ... (1)

$10y - 13x = 3$ ... ... (2)

To eliminate y, multiply (1) by 5, and (2) by 2.

We get

$20y + 45x = -5$

$20y - 26x = 66.$

Subtracting, $71x = -71$

$\therefore \quad x = -1.$

By substitution in (1)

$4y - 9 = -1$

$\therefore \quad 4y = 8. \quad y = 2$

$x = -1, \ y = 2$ is the required solution.

Verification. $4y + 9x = 4.2 + 9(-1) = -1$

$10y - 13x = 10.2 - 13(-1) = 33.$

Method (2).

Ex. 3. Solve: $3x = 7y - 19$ (1)

$31 - 4y = 5x$ (2).

We can eliminate x by substituting in (2) its value in terms of y given by the equation (1).

From (1), $x = \dfrac{7y - 19}{3}.$

Substituting in (2), $31 - 4y = 5 \times \dfrac{(7y - 19)}{3}$

or $93 - 12y = 35y - 95$

$$\therefore -12y-35y=-93-95$$
$$i.e. \quad 47y = 188$$
$$\therefore \quad y=4.$$

By substitution of this value of y in (1), the value of x can be determined

$$3x=28-19=9$$
$$\therefore \quad 3x=3$$
$$\therefore \quad \left.\begin{matrix} x=3 \\ y=4 \end{matrix}\right\} \textit{Answer.}$$

Verification:

Left hand side of (1) $= 3x = 3 \times 3 = 9$

Right hand side $= 7y - 19$
$= 7 \times 4 - 19 = 9.$

Also left hand side of (2) $= 31 - 4y$
$= 31 - 4 \times 4$
$= 31 - 16 = 15$

Right hand side $= 5x = 5 \times 3 = 15.$

3rd Method. By comparison.

Ex. 4. Solve—

$$3x+4y=-18 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$
$$2x-7y=104 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

From (1) $3x=-4y-18$

$$\therefore \quad x=\frac{-(4y+18)}{3} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

From (2) $2x=7y+104$

$$x=\frac{7y+104}{2} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

From (3) and (4)

$$\frac{-(4y+18)}{3} = \frac{7y+104}{2}$$

$\because$ each is equal to x.

Multiply both sides of the equation by 6.

$$-8y-36=21y+312$$

$$\therefore \quad -8y-21y=312+36$$

$$-29y=348$$

$$\therefore \quad y=-12$$

By substitution in (1)

$$3x-48=-18$$

$$\therefore \quad 3x=30, \quad x=10$$

$$\therefore \quad \left.\begin{matrix} x=10 \\ y=-12 \end{matrix}\right\} \text{ is the required solution.}$$

Verification:

$$3x+4y=3\times 10+4\times -12=-18$$

$$2x-7y=2\times 10-7\times -12=104.$$

Ex. 5. $6(x+3y)-(4x+5y)=15$ ... ... (1)

$7(2x-9y)-3(x-2y)=-46$ ... ... (2)

Here the equations are not in the standard form $ax+by=c$ and must be reduced to that form before we can solve them.

From (1) $6x+18y-4x-5y=15$

i.e. $2x+13y=15$ ... ... (3)

From (2) $14x-63y-3x+6y=-46$

i.e. $11x-57y=-46$ ... ... (4)

From (3) $22x+143y=165$

From (4) $22x-114y=-92$

subtracting, $257y=257$

$$\therefore \quad y=1$$

By substitution in (3)

$$2x+13=15$$

$$\therefore \quad 2x=2, \; x=1$$

$$\therefore \quad \left.\begin{matrix} x=1 \\ y=1 \end{matrix}\right\}$$

The student may verify the roots of the equation.

Ex. 6. $\frac{x}{5}-\frac{y-5}{3}=x-2y+1$ ... ... (1)

$$\frac{y}{6}+\frac{10-x}{5}=\frac{x}{15}+\frac{4-y}{2} \quad \ldots \quad \ldots \quad (2)$$

Here also the equations should be cleared of fractions and reduced to the standard form.

Multiplying (1) by 15, $3x-5(y-5)=15x-30y+15$

$$3x-5y+25=15x-30y+15$$

$$3x-15x-5y+30y=15-25$$

$$-12x+25y=-10$$

$$\text{or } 12x-25y=10 \quad \ldots \quad (3)$$

Multiplying (2) by 30

$$5y+60-6x=2x+60-15y$$

$$\therefore \quad -6x-2x+5y+15y=60-60$$

$$\therefore \quad -8x+20y=0$$

$$\text{or } 2x-5y=0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

From (3) and (4)

$$12x-25y=10$$

$$\underline{12x-30y=0}$$

$$5y=10$$

$$\therefore \quad y=2$$

By substitution in (4)

$$2x-10=0$$

$$2x=10$$

$$\therefore \quad x=5.$$

The student may verify the roots.

Exercise X (b).

Solve the following equations by the first method and verify the roots :—

1. $x+y=10$
$x-y=8.$

2. $2x+3y=8$
$2x-3y=2.$

3. $x+2y=5$
$3x+5y=13.$

4. $6x+5y=\cdot 14$
$30x-23y=22.$

Solve the following equations by the second method or the third method and verify the roots :—

5. $x=4y-7$
$10y=7x+13.$

6. $3x-2y=15$
$7x=11y-41.$

7. $5x+1=6(x+1)$
$4x+6=11x+41.$

8. $\frac{x}{2}+\frac{y}{3}=5.$
$\frac{x}{4}+\frac{y}{12}=2.$

9. $\frac{3}{x}+\frac{4}{y}=2$
$\frac{2}{x}-\frac{1}{y}=\frac{5}{12}.$

10. $\frac{3}{x}+4y=8$
$\frac{2}{x}-7y=9.$

11. $\frac{4x+1}{7}-\frac{3x-2y}{5}=\frac{4y-x}{20}$
$\frac{6x-2}{3}-\frac{4y-3x}{6}=\frac{x+5y}{9}.$

12. $6(x-3)+5(y-2)=10(x-y)$
$8(x+4)+9(2y+3)=12(x+2).$

13. $3\cdot 75x-1\cdot 5y=27$
$6\cdot 25x+3\cdot 5y=36.$

14. $5x-2y=13\cdot 3$
$7x+3y=1\cdot 9.$

15. $\frac{x+2y}{3}+1=5$

$\frac{2x-3y}{4}+2=8.$

16. $2x+7y=\frac{4}{7}(y+3)$

$6x+4y=\frac{3}{5}(x-3).$

17. $a_1x+b_1y=c_1$

$a_2x+b_2y=c_2.$

18. $3u+v=a$

$c(u-v)=a-b.$

19. $x:y::a:b$

$x+y:x-y::c:d.$

20. $\frac{y}{x}=\frac{4}{5}$

$\frac{6x+y}{7x+8y}=\frac{9}{10}.$

§ 76. Problems leading to simultaneous equations of two unknown quantities.

Ex. 1. Find a fraction which becomes 1 when 22 is added to its numerator and becomes $\frac{1}{2}$ when 1 is subtracted from its denominator.

Let x be the numerator and y be the denominator.

Then the fraction is $\frac{x}{y}$. The fraction formed when 22 is added to the numerator is $\frac{x+22}{y}$.

$\therefore$ by the question $\frac{x+22}{y}=1$ ... ... (1)

The fraction formed when 1 is subtracted from the denominator is $\frac{x}{y-1}$.

$\therefore$ by the question $\frac{x}{y-1}=\frac{1}{2}$ ... ... (2)

From (1) $x+22=y$

or $x-y=-22$ ... ... (3)

From (2) $2x=y-1$

or $2x-y=-1$ ... ... (4)

By subtraction

$$-x = -21$$

$$\therefore \quad x = 21$$

By substitution in (3)

$$21 + 22 = y$$

or $y = 43$

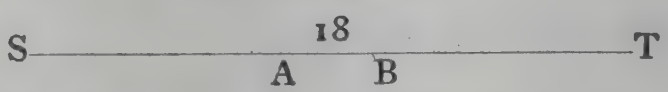
$\therefore$ the fraction $= \frac{21}{43}$.

Verification :—

$$\frac{21+22}{43} = \frac{43}{43} = 1$$

$$\frac{21}{43-1} = \frac{21}{42} = \frac{1}{2}.$$

Ex. 2. A railway train was keeping time when the engine burst. Another engine was obtained in quarter of an hour and the train travelling at three-fourths of its former speed reached the terminus 2 hrs. 30 minutes late. If the accident had occurred 9 miles nearer the terminus with a similar delay of 15 minutes, the train would have been 2 hrs. 15 minutes late. How far from the terminus was the place where the accident happened ?



Let ST be the whole line and let A be the place of accident, and T the terminus ; and let AT be x miles and the usual rate be y miles per hour.

The usual time taken by the train to travel AT at the usual rate is $\frac{x}{y}$ hrs. After the accident, with an initial delay

of $\frac{1}{4}$ hr. the train travelled AT at $\frac{3}{4}$ of its former speed and was late by $2\frac{1}{2}$ hours

$$\therefore \quad \tfrac{1}{4} + \frac{x}{\frac{3}{4}y} = \frac{x}{y} + 2\tfrac{1}{2} \quad \ldots \quad \ldots \quad (1)$$

If the accident happened at B, 9 miles nearer T than A, the train would have travelled BT. $x-9$ miles at $\frac{3}{4}y$ miles per hour and would have been late by $2\frac{1}{4}$ hours

$$\therefore \quad \tfrac{1}{4} + \frac{x-9}{\frac{3}{4}y} = \frac{x-9}{y} + 2\tfrac{1}{4} \quad \ldots \quad \ldots \quad (2)$$

Subtracting (2) from (1)

$$\frac{9}{\frac{3}{4}y} = \frac{9}{y} + \frac{1}{4}$$

$$\frac{4}{3y} = \frac{1}{y} + \frac{1}{36} \quad \therefore \quad \frac{1}{3y} = \frac{1}{36}. \quad \therefore \quad y = 12.$$

By substitution in (1)

$$\tfrac{1}{4} + \frac{x}{9} = \frac{x}{12} + 2\tfrac{1}{2}$$

$$x = 81.$$

AT = 81 miles and the rate of the train was 12 miles per hour.

The student may verify the answer.

Ex. 3. $s = ut - \frac{1}{2}gt^2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

$s = 48$ when $t = 1$; and $s = 64$ when $t = 2$; find u and g and hence find the value of s when $t = 4$.

$48 = u \times 1 - \frac{1}{2}g \times 1 \quad \ldots \quad \ldots \quad (2)$

$64 = u.2 - \frac{1}{2}g.4 \quad \ldots \quad \ldots \quad (3)$

Multiply (2) by 2

$$96 = 2u - g$$
$$\underline{64 = 2u - 2g}$$
$$32 = g$$

Substituting in (2)

$$48 = u - 16$$
$$\therefore \quad u = 64$$
$$\therefore \quad s = 64t - 16t^2$$
$$\therefore \quad \text{when } t = 4,\ s = 64 \times 4 - 16 \times 16$$
$$= 256 - 256$$
$$= 0$$

Ex. 4. In a mile race if A gives B 20 yds. start, A wins by 21 sec., but if A gives B 30 sec. start, B wins by 44 yds. Find the time that each takes to run the mile.

Let x and y seconds be the times taken by A and B to run a mile.

If A gives B a start of 20 yds. B has to run only 1740 yds. to reach the goal while A has to run 1 mile; the time taken by B to run 1740 yds. $= \frac{y}{1760} \times 1740$ sec. and A takes to run 1 mile x sec., and A wins by 21 sec.

$$\therefore \quad x + 21 = \frac{y \times 1740}{1760} = \frac{87}{88}y \quad \ldots \quad \ldots \quad (1)$$

If A gives B 30 seconds start, B is in advance of A by $\frac{1760}{y} \times 30$ yds.

∵ B wins by 44 yds., the time taken by A to run $(1760 - 44)$ yds. is equal to the time taken by B to run

$$\left\{1760 - \frac{1760 \times 30}{y}\right\} \text{ yds.}$$

$$\therefore \quad \frac{(1760 - 44) \times x}{1760} = \frac{y}{1760} \times \left\{1760 - \frac{1760 \times 30}{y}\right\} \quad (2)$$

$$\therefore \quad 1716x = 1760y - 1760 \times 30.$$

Dividing (2) by 44 we have

$$39x = 40y - 1200 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

From (1) $88x - 87y = -21 \times 88$ ... ... (4)

Multiply (4) by 40 and (3) by 87

$$3520x - 3480y = -73920$$

$$\underline{3393x - 3480y = -104400}$$

$$127x = 30480$$

$$x = 240.$$

By substituting in (1)

$$261 = \frac{87}{88}y \; ; \; y = \frac{3 \not{2} \not{1} \times 88}{\not{8} 7} = 264.$$

∴ A takes 4 m. and B takes 4 m. 24 s., to run a mile.

Ex. 5. Between 4 and 5 P.M. I start to the beach for a stroll. I return home between 7 and 8 and find that the hands of my clock have exchanged places. At what time did I start?

Let the distances of the hour hand and minute hand from 12 at the time of starting be x and y minute divisions respectively. Then reckoning from 4 P.M, the minute

hand had moved y minute divisions and the hour hand x—20 minute divisions.

$\therefore\ y : x-20 :: 60 : 5 \quad \ldots \quad \ldots \quad (1)$

[Since the minute hand and hour hand make 1 complete revolution in 60 m. and 5 m. respectively]. On return the distances of the minute hand and the hour hand from 12 were x and y minute divisions respectively. Reckoning from 7 P.M., the minute hand had moved x minute divisions and the hour hand $y - 35$ minute divisions.

$\therefore\ x : y-35 :: 60 : 5 \quad \ldots \quad \ldots \quad \ldots \quad (2)$

From (1) $y : x-20 :: 12 : 1$

$y = 12x-240$

$y - 12x = -240 \quad \ldots \quad (3)$

From (2) $x : y-35 :: 12 : 1$

$x = 12y-420$

$-12y + x = -420 \quad \ldots \quad \ldots \quad (4)$

From (4) $-144y + 12x = -5040$

From (3) $y - 12x = -240$

$\therefore\ -143y = -5280.$

$y = \frac{5280}{143} = 36\frac{12}{13}$

$36\frac{12}{13}$ minutes past 4 is the time of starting.

Ex. 6. The general form of the equation of a straight line is $y = mx + c$. If the points (7, 8) and (3, 4) are on the line, find the values of m and c, and write down the equation to the line.

$y = mx + c$ is satisfied by (7, 8)

$\therefore\ 8 = 7m + c \quad \ldots \quad \ldots \quad \ldots \quad (1)$

It is also satisfied by (3, 4)

$\therefore\ 4 = 3m + c \quad \ldots \quad \ldots \quad \ldots \quad (2)$

By subtraction

$$4 = 4m.$$

$$\therefore m = 1$$

By substitution in (1)

$$8 = 7 + c.$$

$$\therefore c = 1.$$

$$\left.\begin{aligned} m &= 1 \\ c &= 1 \end{aligned}\right\}$$

and the equation to the line is $y = x + 1$.

We have seen that if only one equation connecting the unknowns is given, there are many solutions. In certain problems there are other conditions which create a limit to the number of solutions as will be exemplified in the following problem.

Ex. 7. How many two figure numbers are there so that each of them may be equal to seven times the sum of the digits?

Let x be the fig. in the units' place, y that in the tens' place.

Then the no. $= 10y + x$.

By the question, $10y + x = 7(x + y)$.

$$\therefore 3y = 6x$$

$$y = 2x$$

Give integral values to x beginning from 0

$x = 0, 1, 2, 3, 4, 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = 0, 2, 4, 6, 8, 10 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The values of x and y being digits should be less than 10. Thus we see that the only possible pairs of values of x and y are (1, 2), (2, 4), (3, 6), (4, 8), and the corresponding numbers are 21, 42, 63, 84.

Exercise X (c).

1. If $Ax+By=1$ when A, B are constants; and if when $x=2$, $y=4$; and when $x=3$, $y=5$ find A and B.

2. The sum of two numbers is 6239 and the difference 1761. Find the numbers.

3. Find two numbers such that thrice the first plus 4 times the second is 270; also 5 times the second minus twice the first is 50.

4. Twelve years ago a man was three times as old as his son; eighteen years hence their ages will be in the ratio of 3 : 2; find the man's present age.

5. The contents of a 40 gallons cask would just fill 280 bottles and 5 jars or 240 bottles and 10 jars; find the capacities of a bottle and a jar.

6. If 4 viss of sugar and 5 measures of wheat flour cost Rs. 3-7-0 and 6 viss of sugar and 7 measures of wheat flour cost Rs. 4-7-0, find the cost of a candy of sugar.

7. The wages of 10 men and 12 women amount to Rs. 7-4-0 per day; each man earns 6 as. more per day than each woman, find a man's daily wages.

8. Find two numbers such that one-third of the first and three-fourths of the second make 6; also a fourth of the first and half of the second amount to 48.

9. If A gave B Rs. 5, A would have Rs. 10 more than B. If B gave A Rs. 15, A would have 6 times as much as B. How much has each?

10. A composition of gold and silver weighing 200 oz. is worth £550. But if the proportion of gold and silver be interchanged it would be worth £280 less. The price of an oz. of gold being £3-15*s*., find the price of an oz. of silver and also the proportion of the mixture.

11. Two vessels contain milk and water in the proportion of 3 : 1 and 7 : 3. How should these two kinds be mixed together to give a new mixture containing milk and water in the ratio of 31 : 12?

12. I have a certain sum of money wherewith to buy a certain number of fruits; and I find that if I buy at the rate of 10 for 3 pies,

I shall have to spend 2 as. too much and if at 25 for 3 pies 1 anna too little. How much have I to spend and how many fruits to buy?

13. A person bought a watch for a certain sum and sold it so as to gain 20°/₀. Had he bought it for Rs. 5 more and sold it for Rs 10 more, he would have gained $33\frac{1}{3}$°/₀ on his bargain. What were the cost and selling prices of the watch?

14. A farm is let for a fixed sum and the value of a certain number of quarters of wheat; when wheat is 38*s*. a quarter, the rent amounts to £286; and when it is 56*s*. a quarter, it amounts to £376. Find the fixed sum and also the number of quarters of wheat paid as part of the rent.

15. A ton of ice is manufactured at 6 p. per lb. and sold at 9 p. per lb. A certain quantity is kept for sale at the factory and the remainder is sent to branch shops. If the average loss from melting be $12\frac{1}{2}$°/₀ of the former and 25°/₀ of the latter and if the profit on the ton made is Rs. 17-8-0, find the quantity kept at the factory.

16. Divide £1040 between A and B of ages 18 and 19 respectively so that they may have equal properties at 21, their shares being improved at 8°/₀ compound interest.

17. The area of a certain rectangle would be increased by 2057 sq. ft. if 2 ft. were added to its breadth, and 2 ft. subtracted from its length; and its area would be diminished by ~~1260~~ sq. ft. 804 if 7 ft. were subtracted from its breadth and 2 ft. added to its length. Find the area of the rectangle.

18. A man rode a certain distance at a uniform rate in 4 hrs. If the distance had been 1 mile less and his rate per hour 3 miles more, he would have taken 3 hours. Find his rate and the distance.

19. A man took $5\frac{1}{2}$ hours to travel 21 miles driving for 9 miles and walking the remaining distance. If he had driven only 6 miles, the journey would have taken half-an-hour longer. Find his rates of driving and walking.

20. *M* men and *N* women can do a piece of work in *p* days, *A* men and *B* women can do the same piece of work in *q* days. How long would one man alone take to do the work?

21. A certain sum put out to simple interest amounts to Rs. 708 in 3 years. If the rate of interest be increased by one-third, it will amount to Rs. 744 in the same time. Find the sum and the rate per cent.

22. Two trains 80 ft. and 96 ft. long pass each other when moving on parallel rails in opposite directions in 4 seconds. But when moving in the same direction they pass each other in 36 seconds. Find the rates of the trains.

23. Two trains run in opposite directions at 16 and 20 miles an hour. The faster train passes the slower in 4 seconds and a person in the faster train passes the slower in $2\frac{1}{2}$ seconds. Find the lengths of the two trains.

24. Two trains start simultaneously from T and M at 7-30 A.M. and meet each other at S at 10 A.M. If T is 150 miles from M and if the rates of the trains are as 3 : 5, find (i) the distance from S to M and (ii) the rates of the two trains.

25. A train after travelling for 2 hrs., meets with an accident after which it proceeds at $\frac{4}{5}$ of its former rate and arrives at the terminus $1\frac{1}{2}$ hrs. behind time ; had the accident happened 25 miles farther on, the train would have been $1\frac{1}{4}$ hrs. late. Find the length of the line.

26. A and B run a mile ; at the first heat A gives B a start of 44 yds. and beats him by 51 seconds. At the second heat he gives him a start of $1\frac{1}{4}$ min. and is beaten by 88 yds. In what time can A and B run a mile separately ?

27. In a half-mile race A gives B a start of 22 yds. and beats him by 6 seconds. In a three-quarter mile race A gives B a start of 20 secs. and is beaten by $29\frac{1}{2}$ yds. In what time can each of them run a mile ?

28. Find m and c when it is known that the points (2, 4) and $(-6, 8)$ lie on the line $y = mx + c$.

29. Find the equation in the form $y = mx + c$, of the straight line drawn through the points (4, 5) and (6, 7).

30. Show that there is a single two-figure number which is equal to six times the sum of its digits.

31. Find all the two figure numbers each of which is equal to four times the sum of its digits.

32. A vessel is filled with two kinds of liquids, their volumes being 100 c.c. and 339 c.c. and weighs 800 gms. Again the vessel is filled with the same kinds of liquids and their volumes are 200 c.c. and 239 c.c.; the vessel now weighs 1300 gms. Find the specific gravities of the liquids if the vessel alone weighs 22 gms,

33. The difference of the squares of two numbers is 396 and the difference of the numbers is 2. Find the numbers.

34. In a certain inn, there are a certain number of rooms and a certain number of travellers. If a traveller sleeps in each room, there are a too few rooms. If they sleep two in each room, there are b rooms empty. Find the number of rooms and the number of travellers in the inn.

35. The volume of a given quantity of gas is kept constant. It is known that the pressure varies as the sum of a fixed quantity and the temperature of the gas. When the temperature is 0° C, the pressure is 73 cm. When it is 23°C, the pressure is 82 cm. Find a formula connecting pressure and temperature.

36. $\frac{C}{100} = \frac{F-32}{212} = \frac{R}{90}$.

(1) Convert the following temperatures to the centigrade scale: 70° F, 0° F, 84° R.

(2) Express 32° C in F and R scales.

CHAPTER XI.

MENSURATION OF RECTILINEAL FIGURES.

§ 77. Extension of Pythagoras' Theorem.

Ex. 1. ABC is an obtuse-angled △, B being the obtuse angle; and a, b, c when measured are found to be 2 cm. 4 cm. and 3 cm.

Consider the squares on the sides of the △.

Then $AC^2 = b^2 = 16$ sq. cm.

$AB^2 = c^2 = 9$ sq. cm.

$CB^2 = a^2 = 4$ sq. cm.

$\therefore AC^2 > AB^2 + CB^2$.

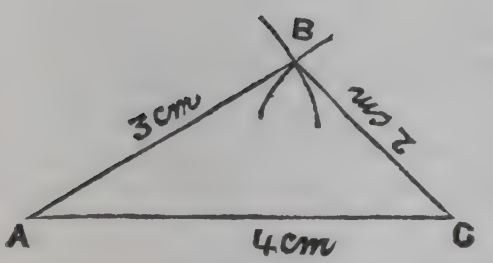


FIG. 82.

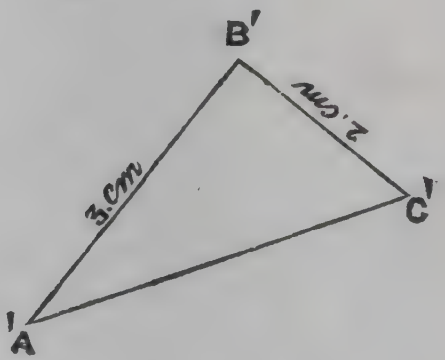


FIG. 83.

Ex. 2. Draw another obtuse-angled △. Measure the sides and test the above property.

Hence learn that **the square on the side opposite the obtuse angle of an obtuse-angled triangle is greater than the squares on the sides containing the obtuse angle.**

Formal proof.—If a triangle A′ B′ C′ (Fig. 83) be constructed with the angle B′ a right angle and having its sides A′ B′ and B′ C′ equal to AB and BC of the triangle ABC of Ex. (1), then the two △s ABC, A′ B′ C′ will have two

sides AB, BC of the one equal to two sides A′ B′ and B′ C′ of the other and the $\angle ABC > \angle A'B'C'$

$\therefore AC > A'C' \therefore AC^2 > A'C'^2$ but $A'C'^2 = A'B'^2 + B'C'^2$.

$\therefore AC^2 > A'B'^2 + B'C'^2$

$> AB^2 + BC^2$.

§ **78.** Thus we have shown that $AC^2 > AB^2 + BC^2$. We can also find the actual difference between AC^2 and $AB^2 + BC^2$ as shown below.

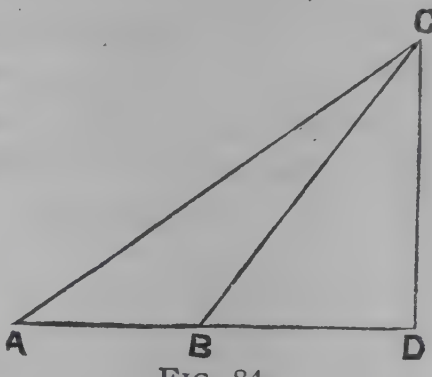


FIG. 84.

Draw CD ⊥ to AB (fig. 84)

Then $AC^2 = AD^2 + DC^2$

$= (AB + BD)^2 + DC^2$

$= AB^2 + BD^2 + 2AB \cdot BD + DC^2$

$= AB^2 + BC^2 + 2AB \cdot BD$

$(\because BD^2 + DC^2 = BC^2)$

Hence we see that **the square on the side opposite the obtuse angle in an abtuse-angled triangle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one side and the projection of the other side upon it.**

[*Note.* BD is the projection of BC upon AB.]

This proposition may be expressed in symbols thus :—

$b^2 = a^2 + c^2 + 2cp$ where a, b, c have the usual meaning and p is the projection of BC upon AB.

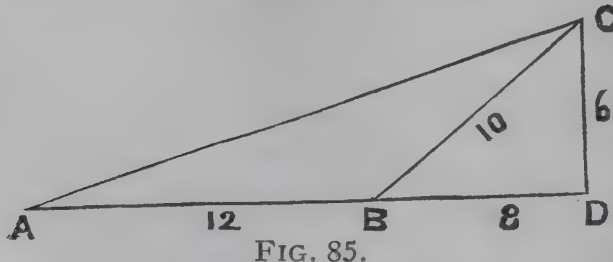


FIG. 85.

Ex. 3. In Fig. 85, ABC is a triangle with an acute angle A; consider the square on the side opposite to it, and the squares on the sides containing it.

$$BC^2 = 10^2 = 100$$
$$AB^2 = 12^2 = 144$$
$$AC^2 = 20^2 + 6^2 = 436$$
and $100 < 144 + 436$
i.e., $BC^2 < AB^2 + AC^2$.

Ex. 4 Construct an acute-angled △. Measure the sides and test the above property.

Hence we learn that **the square on the side opposite an acute angle in any triangle is less than the squares on the sides containing the acute angle.**

We can prove this result, as in the case of the obtuse-angled triangle.

As in the case of the obtuse angle, we can find the difference between BC^2 and $AB^2 + AC^2$.

$\because$ $BD = AD - AB$.
$\therefore$ $BD^2 = (AD - AB)^2$
$= AD^2 + AB^2 - 2AB \cdot AD$.
$\therefore$ $BD^2 + CD^2 = AD^2 + DC^2 + AB^2 - 2AB \cdot AD$,
i.e., $BC^2 = AC^2 + AB^2 - 2AB \cdot AD$.

A similar proof holds good when the ⊥ falls within the △.

Hence we learn '**the square on the side opposite an acute angle in any triangle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by either of the sides and the projection of the other side upon it.**'

[The student is required to express the proposition in symbols.]

Exercise XI (a).

1. Determine whether the following underlined sides are opposite to obtuse or acute angles in the △s whose sides are :—

(*a*) 10, 8, 7.
(*b*) 20, 12, 9.
(*c*) 43, 42, 41,
(*d*) 102, 103, 100.

2. Construct the $\triangle$ in (a); and of the sides containing the angle opopsite to 8, measure the projection of one side upon the other. Verify your result by calculation.

§ 79. Area of a triangle.

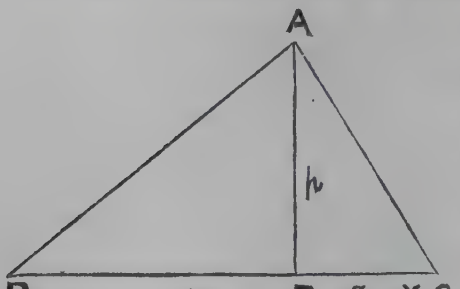


FIG. 86.

It has already been shown that the area of a $\triangle$ is equal to half the product of the base and the altitude. The area can also be computed when the lengths of the three sides are given.

ABC is a $\triangle$ and AD is drawn $\perp$ to BC. Let AD be p and BD be x. (Fig. 86).

Then $CD = a - x$.

AB^2 or $c^2 = p^2 + x^2$ ... ... (1)

AC^2 or $b^2 = p^2 + (a - x)^2$ ... (2)

$$\therefore \quad c^2 - b^2 = x^2 - (a - x)^2$$
$$= 2ax - a^2.$$

$$\therefore \quad \frac{c^2 + a^2 - b^2}{2a} = x \quad \ldots \qquad \ldots \quad (3)$$

From (1) $c^2 = p^2 + x^2$.

$$\therefore \quad p^2 = c^2 - x^2$$
$$= c^2 - \left(\frac{c^2 + a^2 - b^2}{2a} \right)^2$$
$$= \frac{4a^2c^2 - (c^2 + a^2 - b^2)^2}{4a^2}$$
$$= \frac{\{ 2ac + (c^2 + a^2 - b^2) \} \{ 2ac - c^2 - a^2 + b^2 \}}{4a^2}$$
$$= \frac{\{ (c+a)^2 - b^2 \} \{ b^2 - (c-a)^2 \}}{4a^2}$$
$$= \frac{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}{4a^2}$$

$$\therefore p = \frac{\sqrt{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}}{2a}$$

$\therefore$ Area of the $\triangle ABC$

$$= \tfrac{1}{2}\, BC \cdot AD = \tfrac{1}{2}ap$$

$$= \frac{\frac{1}{2}a\sqrt{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}}{2a}$$

$$= \tfrac{1}{4}\sqrt{(a+b+c)(c+a-b)(b+c-a)(b-c+a)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \text{ where } 2s = a+b+c.$$

Note. $(a+b+c)(b+c-a)(c+a-b)(b-c+a)$

$$= 2s(b+c+a-2a)(c+a+b-2b)(b+c+a-2c)$$

$$= 2s(2s-2a)(2s-2b)(2s-2c)$$

$$= 16s(s-a)(s-b)(s-c)$$

$$\therefore \tfrac{1}{4}\sqrt{(a+b+c)(c+a-b)(b+c-a)(b-c+a)}$$

$$= \tfrac{1}{4} \times \sqrt{16s(s-a)(s-b)(s-c)}$$

$$= \tfrac{1}{4} \times 4\sqrt{s(s-a)(s-b)(s-c)}$$

Example.—Find the area of a $\triangle$ whose sides are 30, 28 and 32 units.

Let them be a, b, c respectively so that $a = 30$, $b = 28$, $c = 32$.

Then $2s = a+b+c = 30+28+32 = 90$.

$\therefore \quad s = 45.$

$\therefore \quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{45(45-30)(45-28)(45-32)}$$

$$= \sqrt{45 \cdot 15 \cdot 17 \cdot 13}$$

$$= \sqrt{15^2 \cdot 3 \cdot 17 \cdot 13}$$

$$= 15\sqrt{663}$$

$$= 15 \times 25{\cdot}75 \text{ approximately.}$$

$\therefore$ Area $= 386{\cdot}25$ sq. units approximately.

The student is advised to construct a $\triangle$ of sides 30, 28 and 32 units to a very convenient scale and find the area according to the formula $\triangle = \frac{1}{2}\, a.\, p.$

N.B.—In the case of a right-angled $\triangle$ this formula need not be applied. $\frac{1}{2}$ the product of the sides containing the right angle is the easiest formula for computing the area of a right-angled $\triangle$.

§ **80. The area of an equilateral triangle.** Calculate the area of an equilateral $\triangle ABC$ whose side is x.

Draw AD $\perp$ to BC.

$$AD^2 = AB^2 - BD^2 = x^2 - \frac{x^2}{4} = \frac{3x^2}{4}.$$

$$\therefore \quad AD = \sqrt{3} \times \frac{x}{2}.$$

$$\therefore \quad \triangle ABC = \tfrac{1}{2} BC \,.\, AD = \tfrac{1}{2} x \,.\, \sqrt{3}\frac{x}{2} = \tfrac{1}{4} x^2 . \sqrt{3} \ldots\ldots (1)$$

$$= x^2 \left(\frac{1{\cdot}732}{4} \right) = (\cdot 433) x^2 \ldots\ldots (2)$$

If $x = 5$ inches

$$\triangle ABC = \tfrac{1}{4} \times 25 \times \sqrt{3} = \tfrac{25}{4} \times (1{\cdot}73)$$

$$= \frac{43{\cdot}25}{4} = 10{\cdot}8 \text{ sq. inches.}$$

Verification. According to the formula

$$\triangle = \sqrt{s(s-a)\,(s-b)\,(s-c)}$$

$$= \sqrt{\tfrac{15}{2}(\tfrac{15}{2} - 5)\,(\tfrac{15}{2} - 5)\,(\tfrac{15}{2} - 5)}$$

$$= \sqrt{\frac{15 \times 5 \times 5 \times 5}{16}}$$

$$= \sqrt{\frac{3 \times 5^4}{4^2}}$$

$$= \frac{5^2}{4}\sqrt{3} \text{ (same result as before)}$$

§ **81. Area of a Regular hexagon.** You will find that a regular hexagon (like ABCDEF of Fig. 87) can be divided into 6 equilateral △s (like AOB.)

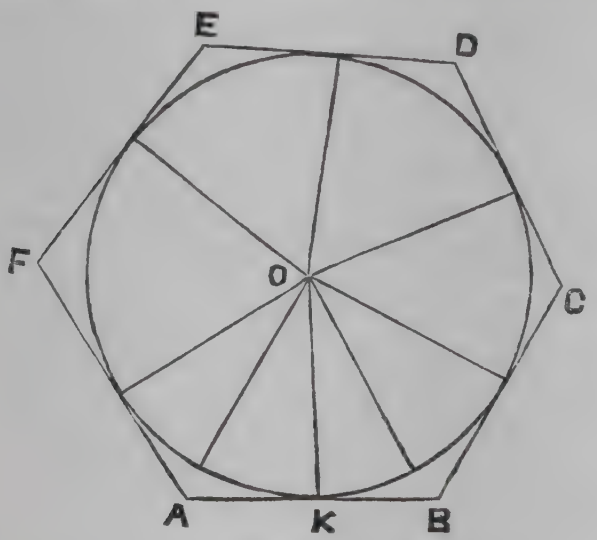


FIG. 87.

∴ the area of the hexagon whose side is x

$$= 6 \times \tfrac{1}{4}x^2 . \sqrt{3} = \tfrac{3}{2}x^2 . \sqrt{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= \frac{3(1{\cdot}732\ldots)}{2}x^2 = \frac{5{\cdot}196}{2}x^2 = 2{\cdot}598x^2$$

Or △AOB = OK . BK = radius of the inscribed ⊙ × $\frac{1}{2}$ the length of the side.

∴ area of a regular hexagon = radius of the inscribed ⊙ multiplied by half the sum of the sides.

N.B.—This applies to any regular polygon. If r be the radius and a be a side of a regular polygon of n sides, the area of the polygon $= \frac{1}{2}\,nra$.

Exercise XI (b).

Apply the formula $\Delta = \sqrt{s(s-a)\,(s-b)\,(s-c)}$ to the calculation (true to four significant figures) of the area of the △ the sides of which are respectively :

1. 100, 112, 116.

2. 204, 216, 224.

3. 39, 52, 65.

4. 83·7, 85, 64.

5. Find the area of the equilateral △s whose sides are 3·5, 4·7, 6·8 and 121 units respectively.

6. Find the area of a regular hexagon having the same sides as in (5).

§ **82.** The formula $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ sometimes leads to tedious multiplication and extraction of square root. The following method may be found preferable in certain cases.

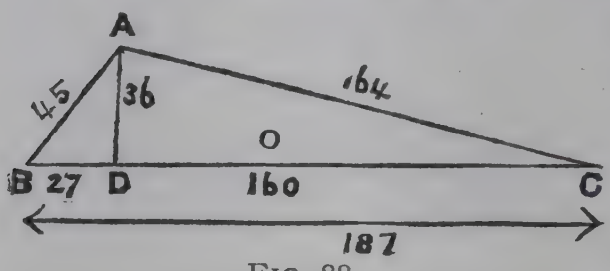


FIG. 88.

Example.—The sides of a triangle are 45, 164 and 187 units of length. Find the area of the triangle.

Let ABC be the △ and let AD be ⊥ to BC, (Fig. 88).

Let O be the middle point of BC.

$AC^2 = AD^2 + CD^2$

$AB^2 = AD^2 + BD^2$

$\therefore AC^2 - AB^2 = CD^2 - BD^2$

$= (CD + BD)(CD - BD)$

$= BC \times 2OD$

$$\therefore OD = \frac{AC^2 - AB^2}{2BC} \quad \ldots \quad \ldots \quad (1)$$

Also $BD = OB - OD$

$= \frac{1}{2}BC - OD \quad \ldots \quad \ldots \quad (2)$

Now $AD^2 = AB^2 - BD^2 \quad \ldots \quad \ldots \quad (3)$

Thus when the sides are given (1) gives OD hence (2) gives BD and hence (3) gives AD.

Then $\triangle ABC = \frac{1}{2}BC.AD$ gives the required area.

Now in this question $OD = \frac{164^2 - 45^2}{2.187}$

$$= \frac{(164 + 45)(164 - 45)}{2.187}$$

$$= \frac{209 \times 119}{2.187} = \frac{133}{2}$$

$$\therefore BD = OB - OD$$

$$= \frac{187}{2} - \frac{133}{2} = \frac{54}{2} = 27$$

and $AD^2 = 45^2 - 27^2$

$$= (45+27)(45-27)$$

$$= 72 \times 18$$

$$= 9^2 \times 4^2 = 36^2$$

$$\therefore AD = 36$$

$$\therefore \triangle ABC = \tfrac{1}{2} BC.AD$$

$$= \frac{1}{2} \times 187 \times 36$$

$$= 3366 \text{ sq. units.}$$

§ 83. Areas of similar triangles.—Let ABC be a △. Take AD so that $AB = k.AD$. (Fig. 89). Draw DE parallel to BC and from A, draw AKL⊥ to DE and BC.

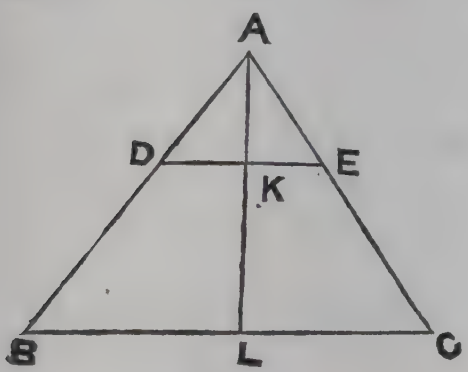


FIG. 89.

Now ∵ $AB = k.AD$

$$\therefore AL = k.AK.$$

and BC is also $k.DE$.

$$\triangle ABC = \tfrac{1}{2} BC.\ AL$$

$$= \tfrac{1}{2}\, k.DE.\ k.AK$$

$$= k^2.\ \tfrac{1}{2}\ DE.\ AK$$

$$= k^2.\ \triangle ADE$$

$$\frac{\triangle ABC}{\triangle ADE} = k^2 = \frac{AB^2}{AD^2}$$

Hence similar △s are to one another as the squares of the corresponding sides.

Example.—ABC is a triangle. (Fig. 90) and AB is 6 inches. BD is cut off equal to one inch. DE is drawn parallel to AC. Compare the area of the quadrilateral ADEC with that of the △ ABC.

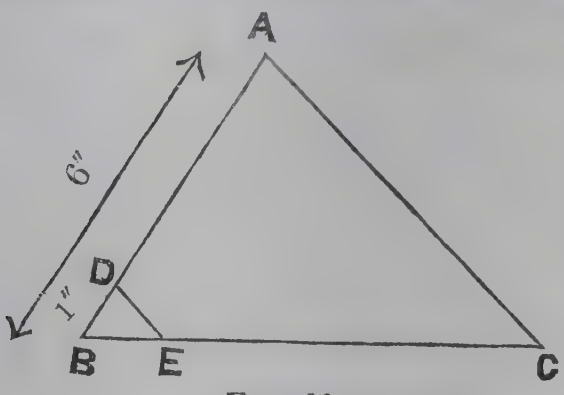


FIG. 90.

ABC and BDE are equiangular and therefore similar.

$$\therefore \frac{\triangle BDE}{\triangle ABC} = \frac{BD^2}{AB^2} = \frac{1^2}{6^2} = \frac{1}{36}$$

$$\therefore \text{the quadrilateral} = \frac{35}{36} \text{ of the } \triangle ABC.$$

Exercise XI (c).

1. Calculate the area of the △ in the following cases by drawing the perpendicular from the vertex on the base and applying the method of Art. 82.

(*a*) 52, 90, 122 units. (*b*) 39, 45, and 42 units.
(*c*) 78, 72, and 165 ,, (*d*) 51, 75, and 84 ,,
(*e*) x, x, x (*f*) x, y, x.

2. ABC is a triangle. D and E are points in AB and AC such that $\frac{AD}{DB} = \frac{AE}{EC}$. Find the ratio in which the △ is divided if $\frac{AD}{DB}$ be (*a*) $\frac{1}{3}$, (*b*) $\frac{2}{5}$, (*c*) $\frac{4}{9}$.

§ 84. Areas of quadrilaterals.—A quadrilateral may be divided into two triangles by drawing a diagonal and the areas of these two triangles may be computed by measurement and calculation thus :—

AE and CF are ⊥s drawn to the diagonal BD of the quadrilateral ABCD. (Fig. 91.)

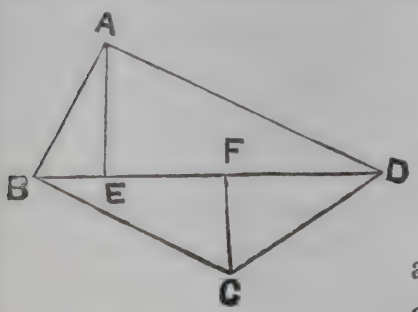


FIG. 91.

Quadrilateral ABCD

$= \triangle ABD + \triangle BCD$

$= \frac{1}{2}BD.\ AE + \frac{1}{2}BD.\ CF$

$= \frac{1}{2}BD\ (AE + CF).$

Hence we learn that the area of a quadrilateral is one half the product of one of the diagonals and the sum of the perpendiculars let fall on that diagonal from the two opposite angular points.

Or the same result may be got thus—draw parallels to the diagonals through the angular points as in Fig. 92.

The quadrilateral ABCD $= \frac{1}{2}$ the $\square^{m} xyzo = \frac{1}{2}yz$, the ⊥ drawn from x on yz which is the sum of the perpendiculars from A and C on BD.

Example.—The diagonal of a quadrilateral is 400 yds. and the ⊥s let fall from the opposite angles are 20 yds. and 30 yds. respectively; find the area.

Area $= \frac{1}{2} \times 400\ (20+30)$

$= 10000$ sq. yds.

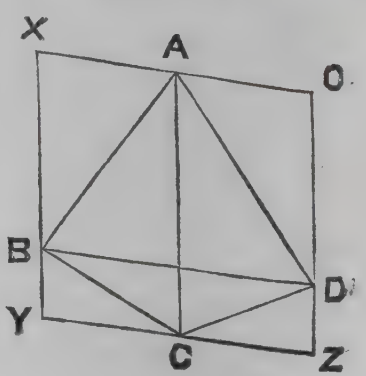


FIG. 92.

§ 85. Area of a trapezium.—A trapezium is a four-sided figure two of whose sides only are parallel. Since a trapezium is a quadrilateral, its area can be determined by using the formula given for a quadrilateral; but hereunder is given an expression for the area of a trapezium in terms of the parallel sides and the distance between them.

Let ABCD be a trapezium having AB and DC parallel.

Join BD. Draw DE ⊥ to AB.

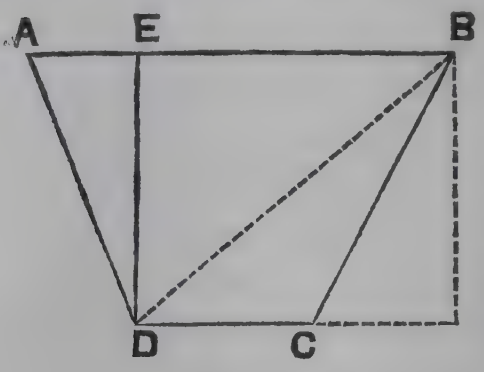


FIG. 93.

Trapezium ABCD
$= \triangle ADB + \triangle DBC$
$= \frac{1}{2}$ AB. DE + $\frac{1}{2}$DC. DE
$= \frac{1}{2}$DE (AB + DC).

The area of a trapezium is one half the sum of the parallel sides multiplied by the perpendicular distance between them.

Example.—The two parallel sides of a trapezium are 1·5 and ·9 inches and the perpendicular distance is ·6 in.; find the area.

Area = $\frac{1}{2}$ the sum of the parallel sides × the perp. distance between them.

$= \frac{1}{2}(1{\cdot}5 + {\cdot}9) \times {\cdot}6 = 1{\cdot}2 \times {\cdot}6 = {\cdot}72$ sq. in.

Exercise XI (d).

1. The area of a trapezium is 10324 sq. ft. The parallel sides are 300 and 152 ft. Find the perpendicular distance between the parallel sides.

2. A piece of land in the form of a quadrilateral ABCD has its diagonal DB = 180 yds. The ⊥s from the angles A and C on the diagonal are 25 yds. and 32 yds. respectively. Find its area.

3. The area of a quadrilateral is 7250 sq. yds. The perpendiculars on a diagonal from the opposite angles are 24 and 26 yds. respectively. Find the length of the diagonal.

4. The two parallel sides of a trapezium are 18·5 ft. and 37·5 ft. The perpendicular distance between them is 8·5 ft. Find the area.

5. Draw the figure to question 4 to scale and verify your answer to the question by using the formula for a quadrilateral.

§ **86. Rectilineal figures.**—The area of a rectilineal figure may be found (1) by dividing it into a number of triangles and using the formula for the area of

a triangle or (2) by reducing the whole figure to an equivalent triangle (*i.e.*, a triangle having the same area as the figure) by geometrical construction and calculating the area of this triangle or (3) by dividing it into a number of trapeziums and right-angled triangles—which method is employed by surveyors in field book.

Method (1). By dividing it into triangles.

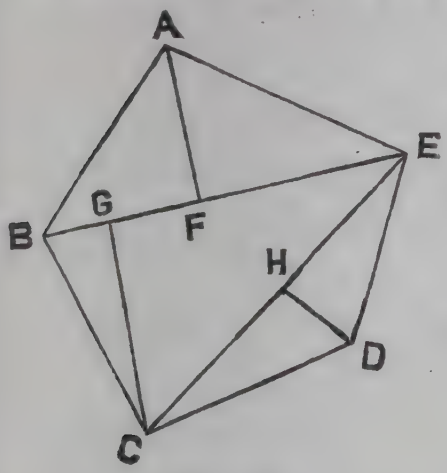


FIG. 94.

Let ABCDE be a pentagon. (Fig. 94). Join BE and CE Draw AF ⊥ to BE. From C draw CG ⊥ to BE and from D draw DH ⊥ to CE.

The pentagon ABCDE $= \triangle ABE + \triangle BCE + \triangle CDE$ $= \frac{1}{2}BE \cdot AF + \frac{1}{2}BE \cdot CG + \frac{1}{2}CE \cdot DH$.

BE, AF, CG and DH and CE may be measured and thus the area of the pentagon ascertained.

Method (2). By reducing it to an equivalent triangle.

Join BE.

Through A draw AA′ parallel to BE meeting CB produced in A′. (In Fig. 95 AA′ is not drawn.)

$\triangle$ ABE $= \triangle$ A′BE because they are on the same base BE and between the same parallels AA′ and BE.

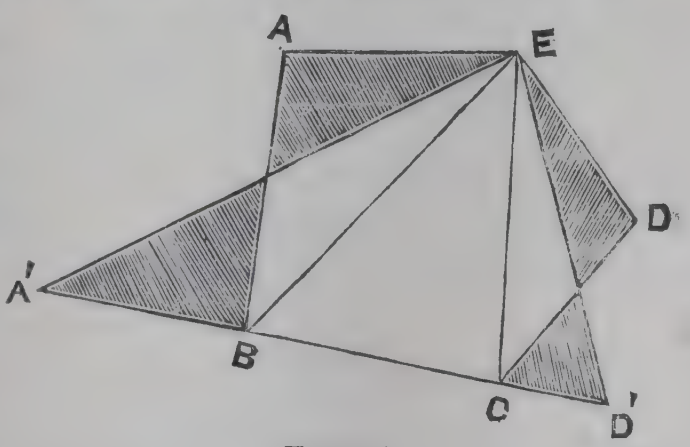


FIG. 95.

$\therefore$ $\triangle$ ABE + the quadrilateral
BCDE = $\triangle$ A′BE + fig. BCDE.

i.e., fig. ABCDE = quadrilateral A′CDE.

Again join CE. Through D draw DD′ parallel to CE meeting BC produced in D′.

$\triangle$ CDE = $\triangle$ CD′E.

$\therefore$ $\triangle$ CDE + $\triangle$ A′CE = $\triangle$ CD′E + $\triangle$ A′CE.

i.e., the quadrilateral A′CDE = $\triangle$ A′D′E.

Thus the pentagon is first reduced to an equivalent quadrilateral which in its turn is reduced to an equivalent triangle.

The area of the $\triangle$ A′D′E may be found by measuring A′D′ and the perpendicular drawn from E on A′D′.

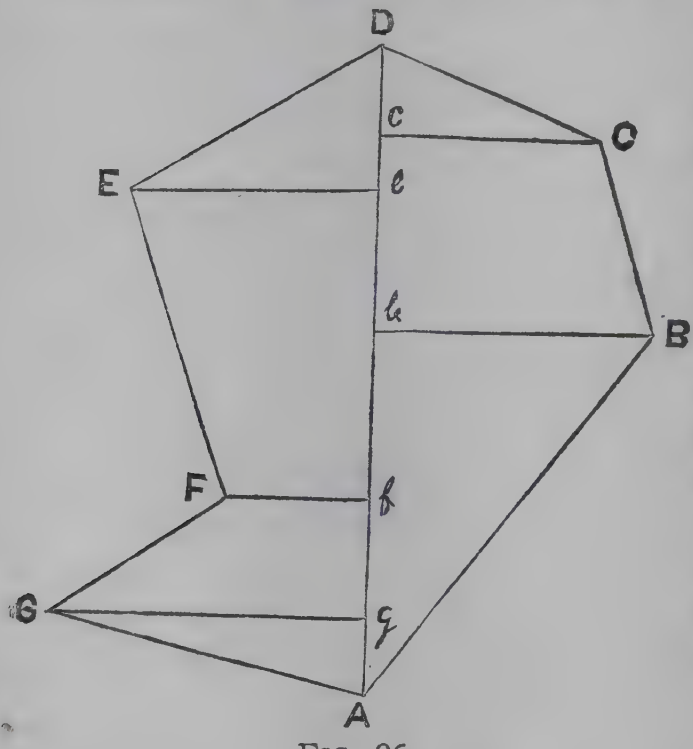


FIG. 96.

Method (3). By dividing the rectilineal figure into a number of trapeziums and right-angled $\triangle$s as shown below.

Find the area of a field of which the adjoining figure is a plan.

The following measurements have been taken :—

AD = 23 yds.
Ag = 3 yds.
Af = 7 yds
$\therefore$ gf = 4 yds.

A b = 13 yds.
A e = 18 yds. $\therefore$ fe = 11 yds.
A c = 20 yds. $\therefore$ bc = 7 yds.
G g = 11 yds.
F f = 5 yds.
b B = 10 yds.
E e = 9 yds.
c C = 8 yds.

Fig. ABCDEFG

= $\triangle$ AGg + $\triangle$ AbB + trapezium G$g$$f$F + trapezium F$fe$E + trapezium bBCc + $\triangle$ DcC + $\triangle$ DEe.

= $\frac{1}{2}$Gg . Ag + $\frac{1}{2}$Ab . bB + $\frac{1}{2}$(Ff + Gg) . fg + $\frac{1}{2}$(Ee+Ff) × ef + $\frac{1}{2}$(cC + bB)cb + $\frac{1}{2}$Dc . cC + $\frac{1}{2}$De . Ec

= $\frac{1}{2}$. 11 . 3 + $\frac{1}{2}$. 13 . 10 + $\frac{1}{2}$ (5 + 11) × 4 + $\frac{1}{2}$ (9 + 5) × 11 + $\frac{1}{2}$ (8 + 10) × 7 + $\frac{1}{2}$. 3 . 8 + $\frac{1}{2}$. 5 . 9

= $\frac{1}{2}$ (33 + 130 + 64 + 154 + 126 + 24 + 45)

= $\frac{1}{2}$ (576) = 288 sq. yds.

§ 87. The above solution leads us to the field book Mensuration of plane figures is very useful because it is employed in the survey of land and in the computation of areas from ascertained measurements. The figure in the last method suggests how a plot of ground with straight lines as boundaries may be divided into right-angled triangles and trapeziums for this purpose and what measurements are likely to be needed.

Measurements along a base line are taken by means of a chain. In the survey of India the 66 ft. Gunter's chain is generally used but sometimes a 100 ft. chain sub-divided into 100 links isalso used.

To find the distance between two points (say two corners of a field) two men called the leader and the driver hold the chain one at each end. The leader stands facing the closing point and the driver at the starting point. The leader drags the chain in the direction of the closing point until the chain is quite straight. They see that it is close to the ground. The leader, who carries with him 10 iron pegs or arrows, then sticks one arrow in the ground at his end of the chain, or makes a × mark if there is no arrow. Then they get up and swing the chain to the side without disturbing the arrow mark. Then both walk in the direction of the closing point until the driver reaches the first arrow. Similarly the measurement of the second chain is made. The leader fixes the arrows as he goes on measuring whereas the driver picks them up as he crosses the points so that the number of arrows in the driver's hand at any time will be a true measure of the chains measured. When 10 arrows are picked up by the driver he hands them over to the leader, so that he may use them for the eleventh chain ; and a pebble or small stone the driver picks up to denote that 10 chains have been measured. The surveyor also makes a note of it.

The surveyors fix the position of the points *g*, *f*, *b*, *e*, *c* from which the offsets, *viz.*, *g* G, *f* F, *b* B, *e* E, *c* C (see Fig. 96) spring by one or the other of the instruments :— the cross staff or the optical square. The latter only is now in use.

§ **88.** The student will now be able to understand how a surveyor records his measurements in the Field Book, regarding a plot of ground surveyed with a single chain line and the offsets from it.

Example 1. Draw a plan and calculate the area of a field from the following notes :—

The Field Book.

	Links.	
	to D	
	1500	
	1250	550 to C
to E 300	1150	
	950	250 to B
to F 900	750	
From	A	go North

The field book should be read upwards, beginning from

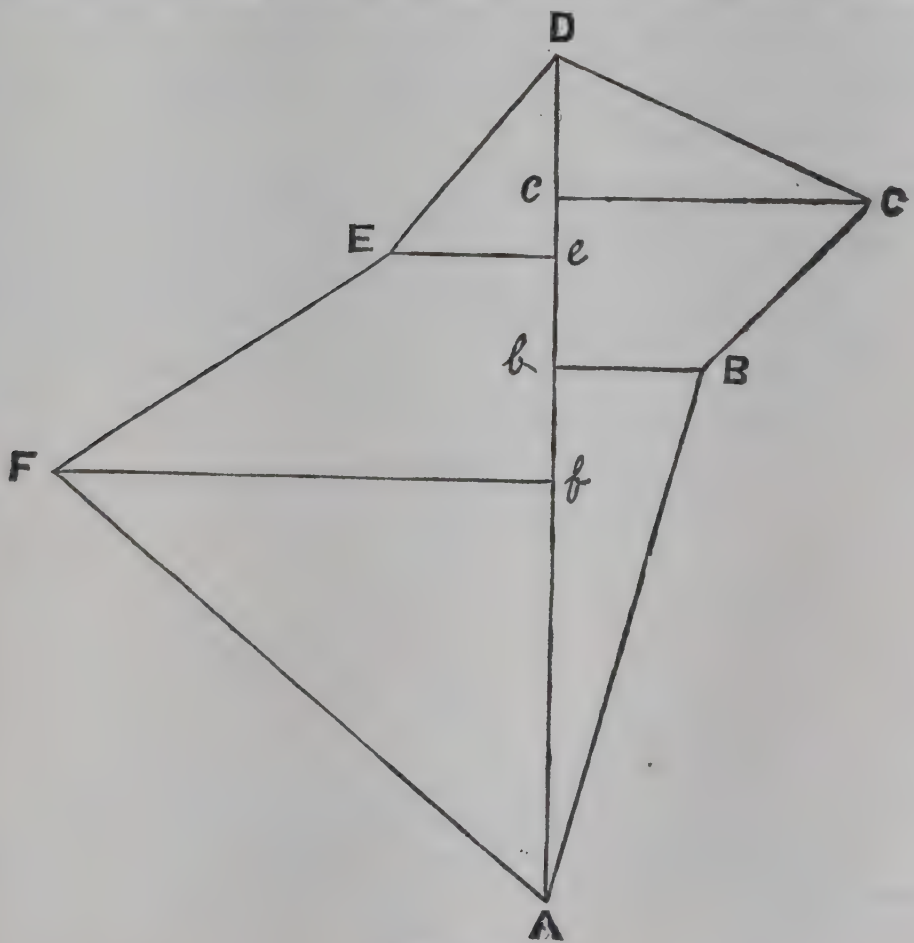


FIG. 97.

the bottom line: 'Go North' tells us that the base or chain line runs from the station A due north. The entries in the middle column refer to measurements made from A along the base line to the point from which the offsets spring. The entries on the right and left hand columns refer to the lengths of the offsets, to the right and left of the chain line.

Now to draw a plan of the field; let the scale be $\frac{1}{10}$ of an inch to 50 links.

First draw AD in a direction due north from A. (Fig. 97.) In AD take a point f so that $Af = 750$ links on this scale. From f draw an offset fF to the left $= 900$ links on this scale. Similarly draw A b and b B; A e, eE; A c, cC; and finally AD. Join AB, BC, CD, DE, EF and FA. The plan is now completed. The area may be calculated as in the preceding example.

§ **89.** Sometimes different chain lines are used, as for example when a field is surveyed by a triangle, and the offsets from each are recorded in the manner already explained.

Example 2. Draw a plan of a field, which is surveyed by a triangle, from the following notes:—

* Links.

	to A	
	2000	
	1800	300
	1600	0
200	1200	
From	C	range to A

* The notes given in this column should be considered as standing above the two columns on the next page so that these are to be read upwards after reading upwards the notes given in those two columns.

	to C	
	2500	
300	1800	
	1500	0
	800	400
From	B	turn left to C
	to B	
	3000	
250	2500	
	2000	0
	1700	300
From	A	go East.

The several horizontal lines refer to the several chain lines.

Thus the field book shows three chain lines AB, BC, CA which must first be drawn ; in other words construct a △ whose sides are 3000, 2500 and 2000 links choosing some

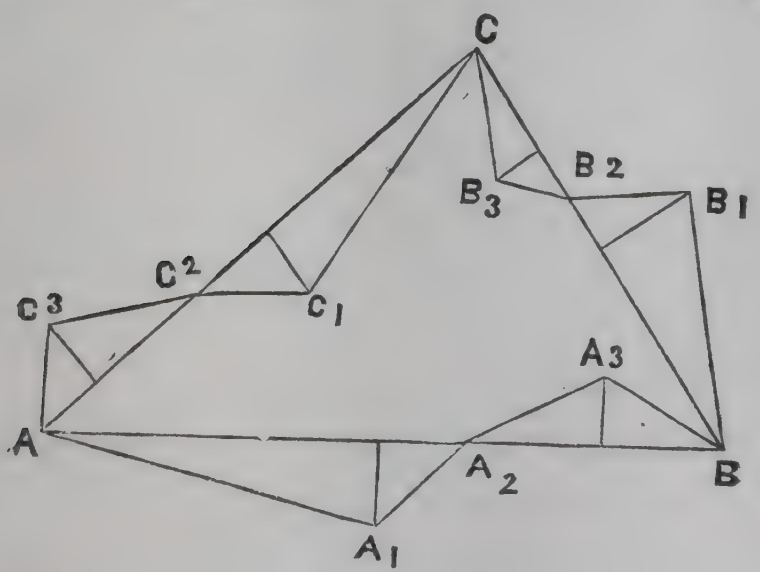


FIG. 98.

scale. The offsets from the chain lines are now to be laid down as explained in the last example.

The area of the figure may be found as follows:—First calculate the area of the $\triangle$ ABC; then the area of all triangles which lie outside the $\triangle$ ABC, *viz.*, $\triangle AA_1A_2$, $\triangle BB_1B_2$ and $\triangle AC_2C_3$; then the area of the $\triangle$s that lie inside the $\triangle$ ABC, *viz.*, $\triangle A_2A_3B$, $\triangle B_2B_3C$, $\triangle CC_1C_2$. It is clear that the area of the field = $\triangle$ ABC + the area of the outside $\triangle$s—the area of the inside $\triangle$s.

The calculation of the area of the figure is left as an exercise to the student.

§ 90. Figures with curved boundaries.

A *a g* G is a figure bounded by a curve line AG and straight lines A *a*, *ag* and *g* G where A *a* and G *g* are $\perp$ to *ag*. We can approximate to the figure A *a g* G by dividing *ag* into equal parts *ab*, *bc*, *cd*.....and drawing ordinates *b* B, *c* C, *d* D etc.....

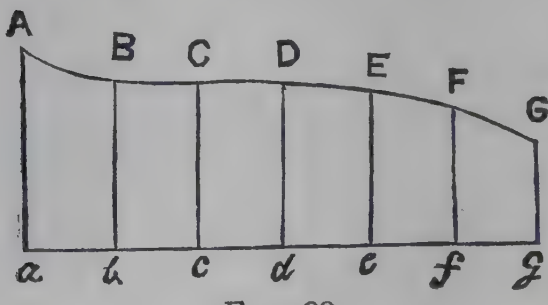


FIG. 99.

If AB is regarded as a straight line, A *ab* B is a trapezium; its area $= \frac{1}{2}$ (A*a* + B*b*) *ab*. Similarly the area of B *b c* C $= \frac{1}{2}$ (B*b* + C*c*) *bc*, and so on.

$\therefore$ the whole area

$= \frac{1}{2}$ (A*a* + 2B*b* + 2C*c* + 2D*d* + ... + G*g*) $\times$ *ab*.

$\because$ (*ab* = *bc* = *cd*.....)

$$= \left(\frac{Aa + Gg}{2} + Bb + Cc + \ldots\ldots + Ff \right) \times ab.$$

Hence we get the rule (known as the Trapezoidal rule). "Add half the sum of the first and last ordinates to the sum of the intermediate ordinates and multiply the result by the common distance between the ordinates."

But the following rule, known as Simpson's *rule*, is sometimes used, as it gives *greater accuracy* in results :*

"Divide the base line into an even number of equal parts so that there are in all an odd number of ordinates. Add together the first ordinate, the last ordinate, twice the sum of the other odd ordinates and four times the sum of all the even ordinates. Multiply the result by one third of the common distance between the ordinates."

Ex. Find the area enclosed by a curve, a straight line and the two end ordinates; given that there are in all 9 ordinates 3, 4, 5, 6, 7, 8, 9, 10, 11, ft. long respectively and that the common distance is 1 ft.

According to Simpson's rule

		4	
	5	6	14
3	7	8	42
11	9	10	112
—	—	—	—
14	21	28	168
	2	4	
	—	—	
	42	112	

$\therefore$ the area $= \frac{1}{3} \times 1 \times 168$

$= 56$ sq. ft.

* The rationale of the rule is beyond the scope of this book.

Exercise XI (e).

Draw a plan and calculate the area of a field from the following notes in each case :—

1.

Links.		
	to D	
	1000	
to E 200	600	
	500	to C 200
	100	to B 150
From	A	go North

2.

Links.		
	to D	
	400	
	300	100 to C
to E 220	260	
	210	80 to B
to F 110	100	
From	A	go N. E.

3.

Links.		
	to E	
	350	
to D 50	300	
	275	100 to F
to C 160	250	
	100	175 to G
to B 150	50	
From	A	go N. W. to E.

4.

Links.		
	to E	
	400	
	300	180 to D
to F 90	250	
	120	75 to C
	110	175 to B
From	A go	N. E. to E.

5.

	Links.	
	to F	
	300	
	275	175 to E
	250	120 to D
to G 100	220	
	180	30 to H
	150	100 to C
	120	100 to B
From A	go	N. W. to F.

6. Draw a plan and calculate the area of a field surveyed by means of a △ ABC, from the following notes :—

	Links.	
	A	
	350	
	250	50
	100	25
From	C	range to A
	to C	
	250	
	125	50
From	B	turn left
	to B	
	500	
	350	25
	275	0
	150	25
From	A	go N. E.

7. Draw a plan of the field ABC and calculate its area from the accompanying field book notes :—

Links.

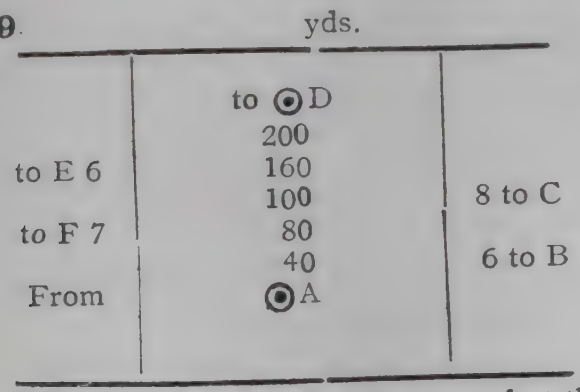
	⊙A	
	500	
30	70	
0	40	
	30	0
	0	20
Turn	to the	left
	⊙C	
	170	0
	40	25
	0	0
	⊙B	
Turn	to the	left
	⊙B	
	540	
	180	
	60	
	0	
From	⊙A	go West

Calculate the area of the field from the following measurments in yards :—

8.

yds.

	to ⊙C	
	70	
to D 8	50	
	32	10 to B
to E 24	25	
From	⊙A	



10. Apply Simpson's rule to find in square feet the areas of the figures having the following dimensions :—

(1) Ordinates 7, 8, 9, 12, 14, 15, 16, common distance 1 ft.

(2) Ordinates 4, 13, 28, 56, 76, 120, 85 ft. common distance 2 ft.

(3) Ordinates 0, 20·6, 25·7, 30·8, 32·3, 45·8, 60 common distance 3 ft.

§ **91. Field work.**—In the first part of this book angles of 'elevation' and 'depression' were defined; and in this chapter an attempt will be made to explain the use of instruments employed to measure angles in vertical and horizontal planes.

Quadrant.—This is really a quarter of a circle. It is fitted with a plumb line and can with advantage be used for the purpose of measuring angles in a vertical plane. P is the object. HOH' is a horizontal line. HOP is the angle of elevation of P as observed from O.

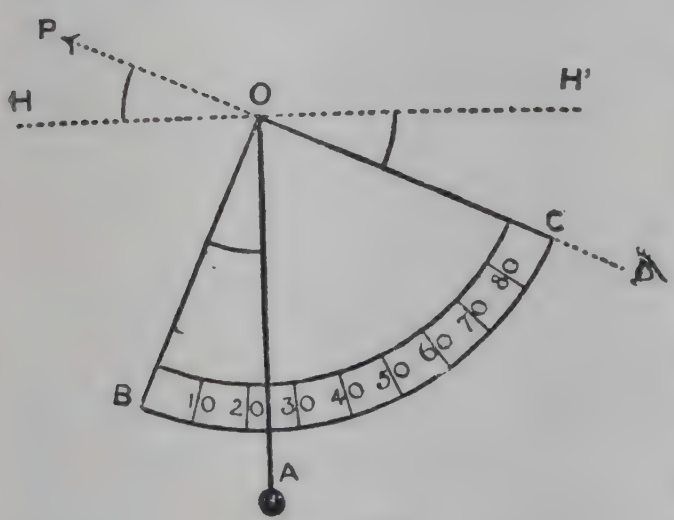


FIG. 100.

$\angle AOH' = 90^\circ = \angle BOC.$

Taking away the $\angle AOC$ from each

we have $\angle BOA = \angle COH' = \angle HOP.$

Hence by reading off the angle $\angle BOA$, the angle of elevation of the object P is found.

The following instructions will be found useful in taking measurements with the instrument :—

(1) Place the plane of the quadrant vertical, with the plumb line swinging freely touching the graduated arc BC.

(2) Look with one eye along the edge CO in the direction of the object whose angle of elevation is required.

(3) Read the angle $\angle BOA$.

The student can easily find out how the quadrant should be used to find an angle of depression.

Exercise XI (f).

1. With a piece of smoked glass before your eye observe the sun's elevation at 9 A.M.

2. Find the angle of elevation of the topmost point of your school from a suitable place in the play ground.

§ 92. The horizontal sighter.—This is used for the purpose of measuring angles in a horizontal plane.

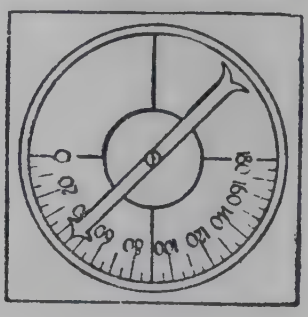


FIG. 101.

To find the angle between the directions of two distant objects in a horizontal plane, we proceed as follows : Having got the apparatus fixed in position on a stand, so that the graduated circle may be in a horizontal plane on a level with the eye, look with one

eye along the middle line of the pointer in the direction of the first object. Take the reading, say 60°. Next turn the pointer so as to point towards the other object and take the reading, say 125°. Then the required angle is 125°—60° or 65°. This is the horizontal angle between the two objects.

Exercise. To find the horizontal distance between two points one of which is inaccessible.

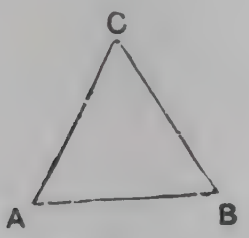


FIG. 102.

Suppose we want to find BC, where C is inaccessible. Take any ground line BA and measure it. Measure ∠BAC and ∠ABC by means of the horizontal sighter. Make a scale drawing of the △ ABC and measure the distance BC.

Exercise XI (g).

1. AB is a straight line and C is a point in AB such that AC is $\frac{1}{3}$ AB. The line turns about C in a clockwise direction through an angle of 60°, and then about A, anti-clockwise through an angle of 65°. Show, by drawing, the final position of AB.

2. From one position a man observes that the angle of elevation of a small chimney is 30° ; at a point 100 yds. nearer, the angle of elevation is 40° ; find the height of the chimney and also its distance from the second point (choose a convenient scale).

3. Two sides of a triangular field are 210 yds. and 240 yds. and the contained angle is 72°. Find the length of the other side and the area of the field.

4. From a point A at a distance of 100 yds. to the south of a tower, its angle of elevation is 35°, and from a point B to the east of the tower the angle of elevation is 30° ; find the distance from A to B.

5. From a certain point the angle of elevation of a tower 300 ft. high is 32°. What is the shortest distance that the observer must walk back so that the angle of elevation may be half as large ?

6. How can you use a quadrant to find roughly the gradient of a sloping road ?

7. A man walks a distance of a mile directly up a slope whose gradient is 1 in 20. What is the increase in his height above sea level.

8. Two ships P and Q are anchored out at sea; from two stations S_1 and S_2 ashore a mile apart, these observations are made:

$\angle S_2S_1P = 80^\circ$, $S_2S_1Q = 40^\circ$,

$\angle S_1S_2P = 52^\circ$ and $\angle S_1S_2Q = 80^\circ$. How far apart are the ships.

9. A balloon is observed from the top and base of a tower 200 ft. high; the two angles of elevation are 40° and 60° respectively. Find the height of the balloon.

10. P and Q are two ships. At noon Q is 15 miles East of P. P is sailing due north at the rate of 4 miles an hour and Q is steaming 40° East of N at 6 miles an hour. How for and in what direction is Q from P at 4 o'clock?

Exercise XI (h).

1. The side of a square is 30 yds. Find the length of its diagonal to the nearest inch.

2. A rectangular field is $1\frac{1}{2}$ miles round and the length of the field is twice its breadth; determine the area in acres.

3. The area of a square field is 4 ac. 3140 sq. yds.; find the cost of erecting a fence round the field at Rs. 2-4-0 per yd.

4. Find the area of a path 6 ft. wide round a rectangular grass plot 60 ft. by 20ft. (Draw a diagram.)

5. The sides of a triangular field are AB = 2800, BC = 3520, CA = 60000 links. It is let at Rs. 20 per acre. Find the annual rent.

6. The co-ordinates of the angular points of a $\triangle$ are (2, 12), (12, −12) (−6, 12). Find the sides and calculate the area by using the formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

7. Find the areas of three equilateral $\triangle$s whose sides respectively are 13, 12, and 5 and show that the former is equal to the sum of the two latter triangles.

8. Inscribe an equilateral Δ in a circle whose radius is 2 inches. Find the area of the Δ.

9. Find expressions for (1) the area (2) the length of a side of a rhombus whose diagonals are a and b. Calculate the area if $a = 12$ in., $b = 5$ in.

10. A rectangle thrice as long as it is wide is inscribed in a circle 6 inches in radius. Find the area of the rectangle.

11. AB and CD are parallel chords of a circle radius 6; AB = 10; CD = 8; find the measure of the area of the trapezium when AB and CD are (1) on the same side; (2) on opposite sides of the centre.

12. The sides of a Δ are 6, 10, 14 inches; find the lengths of the segments into which the perpendicular from the opposite angle divides the longest side.

13. Find the altitude of a triangle whose sides are 14 in. and 18 in. and whose base is 16 in.

14. Find, to the nearest square chain, the acreage of a triangular field whose sides are 30, 36, and 69 chains.

15. ABCD is a quadrilateral figure. AB = 10, BC = 12, CD = 16, DA = 14, BD = 20 inches. Find the area of the figure.

16. Two adjacent sides of a quadrilateral are x and x and the other two sides are y and y. The diagonal dividing it into two isosceles triangles is $2z$; find an expression for the area of the figure.

17. Three equal circles each of radius one inch touch each other. Find the area of the Δ formed by joining their centres.

18. Two parallel sides of a trapezium are 4 and 12 in.; its diagonals are 8 and 10 in. Find the area.

19. The area of an equilateral Δ on a base of 10 inches is approximately 43 square inches. Find the area of an equilateral Δ on a base of 2 inches.

20. The area of a regular hexagon on a side 6 in. is approximately 93·53 sq. inches. Find the area of a similar figure on a base of 3 inches.

21. A field of 36 acres is represented in a plan by a △ whose sides are 50, 34, 24 inches. On what scale is the plan drawn and what length will be represented by 200 inches?

22. In a map, a field of 32 acres is represented by a quadrilateral ABCD. If AC is 5 inches, and the perpendiculars from B and D on AC are 6 and $6\frac{1}{2}$ inches respectively, on what scale was the map drawn?

23. Find the area of a regular pentagon on a side of 5 cm., by reducing it to an equivalent △, giving the result in sq. cm. Check your result by finding the area in any other way.

24. Find by any method the area of a regular decagon on a side of 6 inches.

25. Construct a regular hexagon on a side of 2 in. Show that its area is double the equilateral △ formed by joining the alternate vertices.

26. A quadrilateral figure ABCD has right angles at A and B, and AB = 3 ft., BC = 5 ft., AD = 2 ft. Find its area in square feet.

27. The diagonals AC, BD of a quadrilateral ABCD are at right angles and their lengths are 2 ft. 1 in. and 5 ft. 2 in. Find the area of the quadrilateral.

28. ABCD is a quadrilateral, BC = 20 ft. AD = 26 ft. the diagonal BO = 28 ft. and ∠ C and ∠ A are right angles. Draw a figure to scale and determine the area.

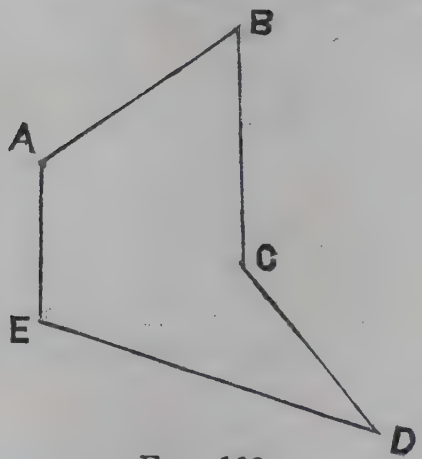


FIG. 103.

29. PQRS is a quadrilateral and QA, SB are drawn at right angles to PR. PA = 6 ft., AR = 10 ft. BS = 4 ft. QA = 12 ft Find the area of the quadrilateral.

30. Find in acres and decimals of an acre, the area of a field which is represented on a scale of 11 ft. to the mile by ABCDEA. (Fig. 103).

31. Figure (104) represents the exact size of a set square. Find the area of the metal, to the nearest tenth of a square inch.

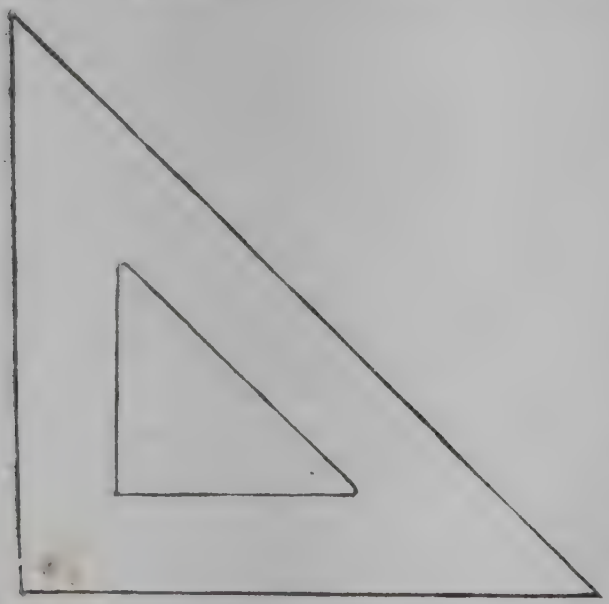


FIG. 104.

32. The legs of a pair of dividers are each 9 cm. and they are opened so that the distance between the end points is 4 cm. If the dividers are placed horizontally on a piece of paper, find the area of the paper between the legs and the line joining the end points.

33. A hall has a gable roof, the extreme section of which is given in the adjoining figure. If the width of the hall is 20 ft. and each of the slant sides of the roof is 12 ft., find the cost of painting the triangular portion of the wall at 6 as. a sq. foot.

FIG. 105.

34. Two equal equilateral triangles are placed one on the other as in figure (106) so that their centres coincide ; find the area of the central hexagon. If this represents a gold leaf cut in the shape of the figure find its value at Rs. 2 per square inch.

35. ABCD is a parallelogram :—adjacent sides are 2·5 inches and 1·6 inches, angle between the sides = 30_0. Find the area of the parallelogram in two ways.

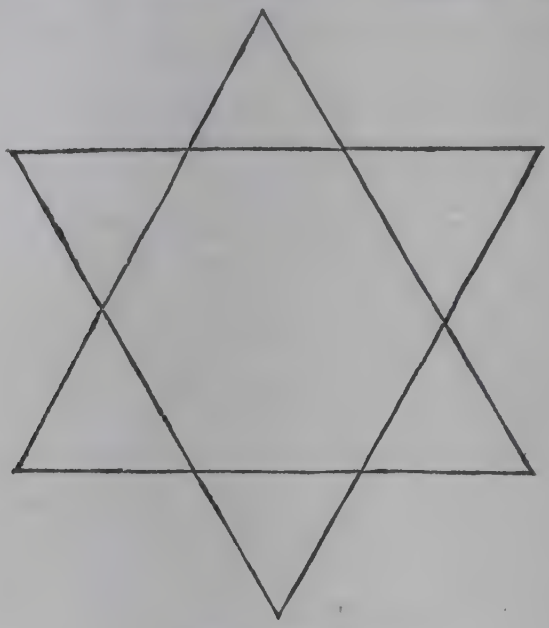


FIG. 106.

36. If a square is equal to the above parallelogram find the side of the square.

37. Find geometrically a square equal to the parallelogram in 35.

38. Construct a rectangle equal to a triangle having its sides 2·3, 1·9 and 1·6 inches.

39. Hence show that it is possible to construct a square equal to any rectilineal figure.

40. An observer in a balloon $\frac{1}{2}$ mile high, observes the angle of depression of a church to be 40°. After ascending vertically for 15 minutes he observes the angle of depression to be 50°. Find the rate of ascent.

41. The angular elevation of a tower at a place A due south of it is 60° and at a place B due west of A at a distance of 50 yds. from it, the elevation is 20°. Find the height of the tower and the distance AB.

42. An object 20 ft. high standing on the top of a tower subtends an angle of 15° at a station distant 100 yds. from the base and in a horizontal line with it. Find the height of the tower.

43. There are two stations lying N. and S. of each other and distant $\frac{1}{2}$ mile. Their angles of depression as observed from a balloon 2000 ft. high are 45° and 60°. Find the bearings of the point of starting of the balloon from the stations assuming the balloon has ascended vertically upwards.

44. A man ascends a mountain by a direct course, the inclination of his path to the horizon being at first 30^0 and afterwards changing suddenly to 45^0 which continues to the summit. Given that the mountain is 1500 ft. high and the angle of depression of the starting point as observed from the summit is 50^0, find the length of the ascent.

45. H is a hill top, P_1 and P_2 two points at sea level 2000 ft. apart; it is found that $\angle HP_1P_2 = 54^0$ and $\angle HP_2P_1 = 80^0$ while the elevation of the hill top as observed from P_1 is 15^0. By careful drawing, find the height of the hill above the sea level.

REVISION PAPERS—II SERIES.

1.

1. A rectangular floor 40 ft. by 24 ft. is to be paved with hexagonal tiles each side being 4 inches. How many whole tiles will be required, and also how many triangular pieces will be required to fill up the remaining spaces at tbe edges?

2. Calculate the velocity of wind blowing directly on a surface with a force P according to the formula $P = 0{\cdot}0023 \cdot V^2$ where P is the pressure in weight per sq. ft. and V is the velocity in feet per second, when (1) $P = 2$ and (2) $P = 12$.

3. Find the amount at compound interest of £450 in 2 yrs. at 4 % per annum, interest payable half-yearly.

4. The sides of a triangle are, 96, 104 and 40 units. Investigate what kind of a triangle it is and show that the radius of the circumscribed circle is 52 units Give reasons for your answer.

5. In a quadrilateral ABCD, AB = 5 cm., BC = 6 cm., AC = 6·5 cm. $\angle ADC = 30^0$ and AD = 3·4 cm. Construct the quadrilateral and find its area.

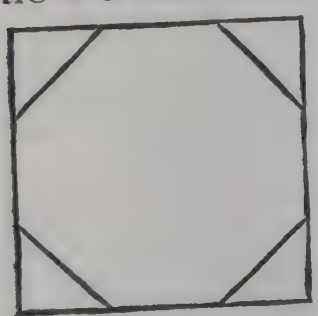


FIG. 107.

6. Find graphically the square root of 12.

7. ABCD is a square whose side is 2 inches. From the corners four triangular strips are cut off so that the figure formed is a regular octagon. Calculate the length of a side of the octagon.

8. (*a*) Solve graphically :—

$$2x + 3y = 7 \quad \ldots \quad \ldots \quad (1)$$

$$4x - 7y = 1 \quad \ldots \quad \ldots \quad (2)$$

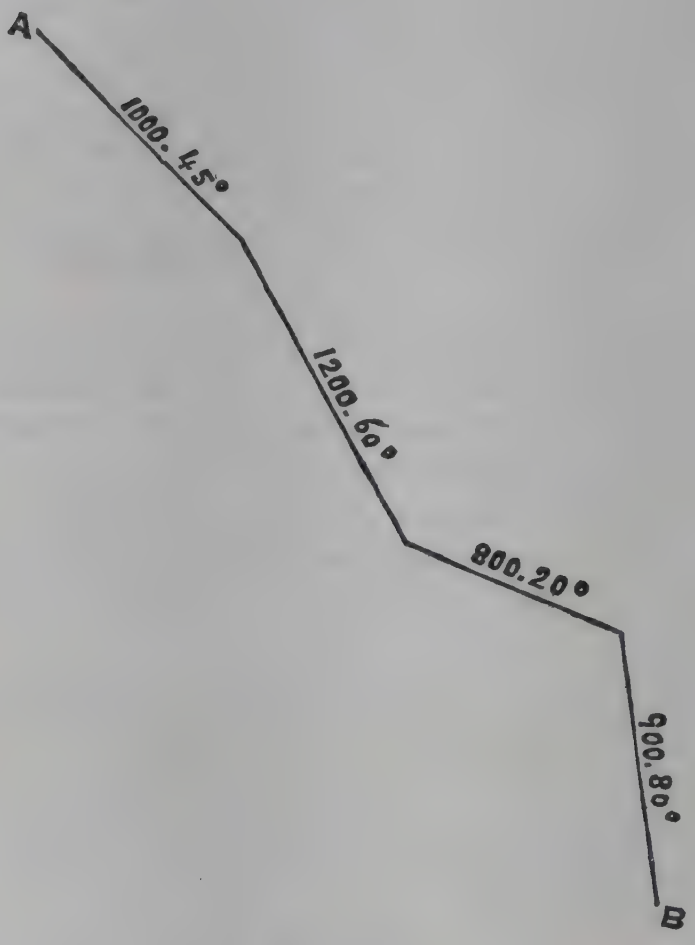


FIG. 108.

(*b*) A survey line AB is run as shown in the sketch. The length of each part is given in links. The angle written against each part is its inclination to the horizon. Find the direction and distance of B from A.

2.

1. Two metal plates of the same substance are in the form of regular decagons. Their sides are in the ratio of 3 : 2. If the first weighs 6·2 oz., find the weight of the second.

2. ABC is a triangle. O is the circumcentre of the Δ and OD is $\perp$ to BC. Express the angles of the Δ BOC in terms of the angle A of the ΔABC (1) when O falls within the Δ; (2) when O falls without the Δ.

3. The hypotenuse and one side of a right-angled Δ are 250169 and 13000 units respectively. Find the length of the other side.

4. Assuming that a gold jewel worn by a person suffers diminution in weight and that the weight at the end of a year is always 96·5 % of that at the beginning of the year, find the weight of a pair of gold bangles of 13½ pagodas weight after 4 years.

FIG. 109.

5. The lower part of the door of a brougham is in the shape shown in the figure drawn to scale:—(1 inch to 2 ft.) Divide the width of the door into 5 equal parts and find the area by drawing the ordinates.

6. The field ABCD for which the following measurements (given in links) were taken is bounded by straight lines. Draw a plan on the scale of 1 inch to the chain and calculate the area of the field.

	D	
	100	
	200	120 E
C 80	150	
B 90	75	
	70	60 to F
	⊙A	

7. A passenger steamer started from Colombo to Bombay and maintained a uniform speed for 36 hours, when her engines broke down, causing a delay of 4 hours. After repairs, she proceeded on her journey at 3½ miles per hour less than before and consequently

arrived at Bombay 25 hours late. If she had left Colombo at the reduced rate of speed and proceeded uniformly without the accident, she would have arrived 3 hours later than she did. Find the distance between the two ports.

8. (*a*) Simplify $\frac{3-\sqrt{7}}{3+\sqrt{7}}$ to 4 significant figures.

(*b*) Find the fraction such that if 7 be added to the numerator, its value will be $\frac{3}{5}$ but if 3 be subtracted from the denominator, the value may be $\frac{1}{2}$.

3.

1. ABC is a triangular metal plate and weighs a grain. The sides AB and AC are trisected at D and E and the △ ADE is cut off from the plate; find the weight of (1) the remaining portion of the plate when D and E are (1) nearer A than B and C (2) nearer B and C than A.

2. ABC is a triangle inscribed in a circle. x, y and z are the middle points of the arcs BC, CA, AB opposite to the angles of the △. Express the angles of the △ xyz in terms of the angles of the △ ABC. Measure the angles and verify your answer.

3. The diagonal BD of a quadrilateral garden ABCD is 1000 links. The perpendiculars on BD from A and C are respectively 375 and 652 links: find the area in acres and cents.

4. The value of a piece of machinery depreciated yearly to 95 % of the value at the beginning of the year. The cost price was Rs. 3250. What was its value at the end of the fifth year?

5. ABC is a triangle. O is the middle point of BC. AD is ⊥ to BC. Applying the formula,

$(x+y)^2+(x-y)^2=2x^2+2y^2$

prove that

$$BD^2+CD^2=2BO^2+2OD^2$$

and hence prove that

$$AB^2+AC^2=2BO^2+2AO^2.$$

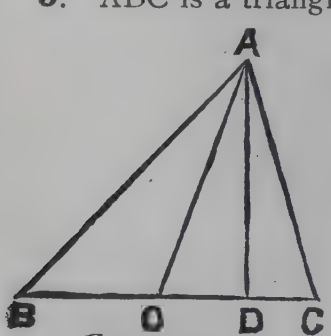


FIG. 110.

6. The sides of a triangle are 6, 7, and 8 inches. Calculate from the above relation the lengths of the medians of the Δ, correct to the first decimal place.

7. A steamer P is in danger and stationary. She fires a gun which is heard in another steamer Q approaching her. P fires another gun 12 minutes after the first and this is heard 10 min. 40 sec. after the first. At this moment Q is 3 miles from P. How soon will the two steamers be along side of each other? Answer to the nearest second. (Velocity of sound = 1100 ft. per second).

8. (*a*) Eliminate t from the equations.

$s = ut + \frac{1}{2}ft^2$ and $v = u + ft$.

(*b*) AD is a tower with a flag staff DE and ABC is a horizontal line (BC = 50 ft.) through the foot of the tower. $\angle ABD = 25^\circ$, $\angle ACD = 16^\circ$, $\angle ACE = 30^\circ$. Draw this figure to scale and measure AD and DE.

4.

1. Find the quantity of earth work in an embankment 900 ft. long, of which the breadth is equal to 20 ft. at the top, the slope on one side is 4 to 1 and on the other $1\frac{3}{4}$ to 1 and the height of the embankment 16 ft.

2. Find the area of a square inscribed in a quadrant whose radius is 2 inches, two sides of the square being along the radii.

3. A balloon at an altitude of 4200 ft. is 4825 ft. distant from the point from which it ascended: how far has it been drifted by the wind?

4. Find the compound interest on Rs. 600 for three years at $4\frac{1}{2}\%$.

5. The expression $A(x^2+y^2+z^2)+B(xy+xz+yz)$ has the value 30, when $x = 1, y = 1, z = 1$,

and has the value 10, when $x = 1, y = 0, z = 0$,

Find A and B.

6. The parallel sides of a trapezium are respectively 9 ft. and 15 ft. Two straight lines are drawn across the figure parallel to these so that the four are at equal intervals; find the lengths of the straight lines and compare the areas of the strips.

7. Draw, as accurately as you can, a fair sized plan of a field ABCDEF from the following notes. Then find the area in acres, and sq. chains.

Links.

	D	
	3250	
	2650	950 to C
to E 400	2200	
	2050	500 to B
to F 1800	2000	
From	A	go North.

8. A rectangular area, the length being 5 times the breadth, contains 17500 acres. How many miles of fencing would be necessary to enclose it?

5.

1. ABC is a triangle. The in-circle touches the sides of the △ in D, E, F. Express the angles of the △ DEF in terms of the angles of the △ ABC.

2. The area of a rhombus is 1416576 square inches and one diagonal is 1344 inches; find the other diagonal and also the length of a side.

3. A capitalist having a sum of money to invest, places one-half in a bank at 4½ per cent. compound interest added yearly to the deposit; and the other half in a bank at 4 % compound interest but the additions of interest are made every six months. At the end of two years he finds there is a difference of Rs. 499-10-0 in the amounts to his credit in the two banks. Find the sum originally deposited.

4. A train going at the rate of 36 miles an hour overtakes another train 176 yds, long, going in the same direction on a parallel line at the rate of 27 miles an hour, and completely passes it in 45 seconds: find the time in which the trains would have completely passed one another, if they had been going in opposite directions.

5. The sides AB, BC, CD, DE, EA of a pentagon ABCDE are 3 ft., 4 ft., 5 ft., 6 ft. and 3 feet respectively and the ∠s A and E are 120° and 90°. Find the area in square feet to the nearest tenth.

and check your result by drawing the figure on squared paper and counting squares.

6. Show that the straight lines represented by the equations $y=2x+4$, $y=4x-2$ and $2y=7x-1$ all pass through a point.

7. A and B run a mile race and A wins by 40 yds.; afterwards B and C run a mile race and B wins by 40 yds. If A and C run a mile race, by how much would A win, it being assumed that each runs at the same rate as before? (Draw a graph and illustrate your answer).

8. Extract the square root of 13 to four decimal places, one-half to be worked by the ordinary method of extraction and the other half by contracted division.

6.

1. ABC is a triangle. D, E, F are the feet of the $\perp$ s drawn from A, B, and C respectively on the opposite sides. Express the angles of the $\triangle$ DEF in terms of the angles of the $\triangle$ ABC.

2. ABC is an equilateral triangle inscribed in a circle, and D is the middle point of the arc opposite to the angle A. Calculate the magnitude of the angle DBC and hence show that AD passes through the centre.

3. Solve :—

$$x+a : y-b :: x+b : y+a$$
$$2x-y : y+b :: 3 : 1.$$

4. A and B started together to run a race. At the end of 4 minutes when A has run 800 yds. at a uniform pace, and was 80 yds, ahead, he slackened his speed by 30 yds. a minute and thus reached the winning post 1 minute after B who ran uniformly throughout. How long did the race last?

5. Find the exact difference between the simple and compound interest on Rs. 6,000 for 3 years at $3\frac{1}{2}$ % per annum.

6. The sides of a triangle are 34, 30 and 16 inches respectively. Find the length of the median bisecting the side 34 inches and also find the cosines of the three angles of the triangle.

7. The time of swing of a pendulum is proportional to the square root of its length. If when the pendulum is 1 metre long

its time of swing is 1 sec., what must be its length so that the time of swing may be $\frac{1}{4}$ sec.? What will be the time of swing of a pendulum 9 metres long?

8. Evaluate

(*a*) $\dfrac{8+\sqrt{3}}{8-\sqrt{3}} - \dfrac{8-\sqrt{3}}{8+\sqrt{3}}$.

(*b*) P_1 and P_2, two points 300 metres apart, are in the same vertical plane with a hill top H above the plain. The angles of elevation of H at P_1 and P_2 are measured and found to be 25° and 40°. Find the height of the hill top, by drawing to a suitable scale.

7.

1. The diagonals of a parallelogram are 36 ft. and 26 ft. respectively and one side is 24 ft. Find its area.

2. An equilateral triangle and a square have the same area. Compare their perimeters.

3. A person invests Rs. 1,000 in the Indian Bank at 5 per cent. interest payable yearly; at the end of every year he withdraws Rs. 200. When will his money transactions with the bank come to a close?

4. Solve the equations—

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2.$$

5. P, Q, R are three light houses along a straight coast PQR. PQ = 5 miles and QR = 4 miles. A ship S is off the coast and the angles PSQ and QSR are 60° and 72°. Construct the position of the ship, and find the distance of the ship from the coast.

6. The chord of an arc is 24 ft. and the chord of half the arc is 15 ft. Find the diameter ot the circle.

7. Plot the points (6, 0), (10, 0), (10, 16), (6, 10). Join the points in the order given. What kind of figure is formed? Find its area.

8. *A*B and CD are two vertical poles. AB is 40 feet high and CD is 30 feet high and the distance on the ground between the

posts is 25 feet. Find (*a*) the distance between the tops of the posts (*b*) the area of the figure enclosed by ABDC.

8.

1. (*a*) A, B and C are three points on the circumference of a circle. O is a point within the circle. Given that $\angle AOC = 2\angle ABC$, can you infer that O is the centre? Give reasons for your answer.

(*b*) Given that O is the centre of a circle, that A and C are points on the circumference and that the $\angle AOC = 2\angle ABC$. Can you infer that B is a point on the circumference? Give reasons for your answer.

2. Two numbers are in the proportion of $\frac{4}{13} : \frac{1}{22}$. When they are increased by 6 and 5 they are in the proportion of $1\frac{1}{3} : \frac{3}{4}$. Find the numbers.

3. The length of a pendulum is 36·5 in. The distance between the two extreme positions of the bob is 2·5 in. Find the distance of the line joining the extreme positions of the bob from the point of suspension.

4. From the formula $A = P\left(1 + \frac{r}{100}\right)^n$ find the value of

(*a*) A, when $P = 200$, $r = 6$, $n = 4$

(*b*) P, when $A = 441$, $r = 5$, $n = 2$.

5. Upon the line joining the points (20, 0) and (30, 0) as base, describe a regular hexagon and write down the co-ordinates of all the angular points.

6. The number of sides of a regular polygon is two more than that of another; the difference between the magnitudes of a pair of the interior angles of the figures is 12°. Find the number of sides of each polygon.

7. The strength of a beam of wood is proportional to the square of its thickness, other things being the same. If a beam 1 foot thick will bear a weight of 1 ton. 4 cwt, what must be the least thickness of a beam which will bear a weight of 6 tons?

8. The length of the arc of a meridian from the equator to the North Pole is 6200 miles. Assuming the latitude of Madras to be 13° North, find the distance of Madras from the North Pole and from the equator. (Assume the earth to be a sphere.)

9.

1. ABCD is a trapezium. The parallel sides are 12 and 8 and the non-parallel sides 7 and 6. Find the distance between the parallel sides. Compare the area of the trapezium with that of the whole triangle formed by producing the non-parallel sides to meet.

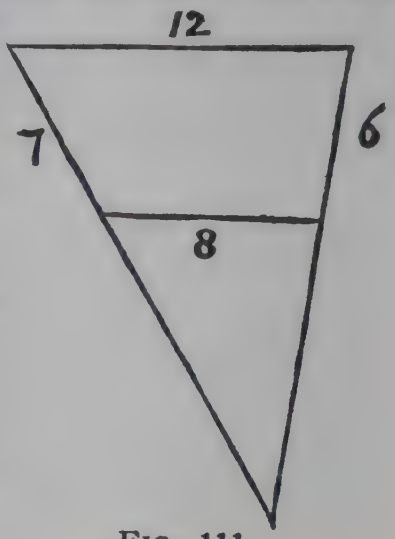


FIG. 111.

2. Being given that the earth turns ronnd once in 24 hours, find how long it will take to turn through 80°. Find the time in Greenwich when it is 12 o'clock noon at Madras, given that the longitude of Madras is 80° E.

3. Plot the points (12, 0), (12, 24), (26, 40), (50, 40), (50, 0). Join the points in order and find the area of the figure thus formed. Suppose that this is drawn to scale $(\frac{1}{200})$ find the area of the plot of ground in square feet.

4. Solve graphically the equation

$$\frac{2{\cdot}59}{6{\cdot}83} = \frac{x}{3{\cdot}84}.$$

N.B.—Use your diagonal scale.

5. Given that

$$W = Fs \ldots\ldots (1)$$
$$F = mf \ldots\ldots (2)$$
$$V^2 = 2fs \ldots\ldots (3)$$

Elminate F and s, from the above relations.

6. Draw a rough sketch of the field ABC aud calculate its area from the accompanying field book notes.

	⊙A	
	500	
25	75	
0	35	0
	25	15
Turn	⊙C	to the left
	⊙C	
	175	0
	45	40
Turn	⊙B	to the left
	⊙B	
	550	0
	175	40
	60	15
	0	0
From	⊙A	go East

7. The electrical resistance of a wire of circular section is inversely proportional to the square of the radius of the section. If the resistance of a wire is 1 when $r = 0{\cdot}06$, find the length of the radius when the resistance is 50.

8. ABC is a $\triangle$, the $\triangle A_1 B_1 C_1$ is formed by joining the middle points of the sides opposite to A, B and C. Similarly the $\triangle A_2 B_2 C_2$ is obtained from the triangle $A_1 B_1 C_1$ and so on. Express the area of the $\triangle A_4 B_4 C_4$ in terms of that of the $\triangle$ABC.

10.

1. The parallel sides of a trapezium are 110 ft. and 154 ft. and the other sides are 50 ft. and 62 ft. Find the area.

2. The span of a bridge the form of which is an arc of a circle is 192 ft. and the height of the arch is 24 ft.; find the radius.

3. $T^2 = S_1 S_2$; where T is the length of a tangent drawn to a circle from a point without it and S_1 and S_2 are the segments of a secant drawn from the same point. Use this property to extract the square root of 12.

4. AB is 48 cm. and $\angle C = 30°$. Calculate the maximum area of the $\triangle$ ABC.

5. Solve :—

$$\frac{x}{y} = \frac{21}{12} \quad \ldots \qquad \ldots \quad (1)$$

$$\frac{8x+9y}{10x-9y} = 30\tfrac{1}{2} \quad \ldots \qquad \ldots \quad (2)$$

6. Find the equation of a line passing through the point (a, b) and (c, d).

7. The following rule for finding the distance of the horizon is very nearly correct. "Take the number of feet in the height of the station above the sea level and increase it by half that number; the square root of this quantity will give the distance of the horizon in miles." Hence find, to one decimal place, the distance of the horizon from the top of Kanchenjunga whose height is 28,156 ft. Find also the height of another mountain peak 350 miles from Kanchenjunga in order that its top may be just visible from that of the latter.

8. The following table shows the amounts of, £ 1 at compound interest at 4·5% for different periods.

Years.	0	5	10	15	20	25	30	35	40	45	50
A.	1	1·25	1·55	1·93	2·41	3·00	3·74	4·66	5·80	7·25	9·03

Draw a graph to show the amounts for different periods and find when a sum will double itself at this rate.

CHAPTER XII.

CIRCLE, CYLINDER, PRISM.

§ 93. To find the length of a curved line.

All curved lines may be regarded as made up of a very large number of straight lines. If an infinite number of such small straight lines could be taken, the sum of their lengths would be equal to the curved line.

A pair of dividers may be opened to a small given distance, say, 4 mm. One end of the dividers may be placed on one extremity of the curve, and the other end on the curve. The dividers may then be rotated on the second leg until the end of the first leg is on the curve and beyond the second leg. The process may be repeated until the whole curve is measured. This process is known as stepping off with the dividers (*vide* Fig. 112). Then the length of the curved line is equal to 4 mm. (the distance between the ends of the dividers) to the number of steps taken with the divider.

FIG. 112.

Or *by means* of cotton thread.

Place one end of a piece of cotton thread at one extremity of the curve. The thread may be made to coincide with a small portion of the curve, the next portion of the thread with the next portion of the curve and so on (*vide* Fig. 113). The length of the whole thread used may be measured.

FIG. 113.

Similarly a strip of tracing paper on which a straight line has been drawn may be used instead of the cotton thread.

§ **94. Circumference of a circle.** It may be shown, by measuring the circumferences and diameters of different circles, as indicated in the Chapter on Compound Proportion, that the circumference of a circle varies as the diameter; the constant ratio of the circumference to the diameter is generally denoted by π; thus if the diameter of a circle is d, its circumference is πd.

Also if the radius is r, the circumference is $2\pi r$.

It is also found from experiment that π is approximately equal to 3·1416 (to 4 decimal places) or 3·142 (to 3 places) or $3\frac{1}{7}$.

Example. The diameter of a circle is 16·5 inches; find its circumference. (Assume $\pi = 3{\cdot}1416$)

Circumference $= 3{\cdot}1416 \times 16{\cdot}5$ in.

It is enough if the answer is got correct to one decimal place, *i.e.*, to 3 significant figures.

$$\begin{array}{r} 3142 \\ \underline{\quad 165} \\ 3142 \\ 18852 \\ \underline{15710} \\ 518430 \end{array}$$

51·8 in. *Ans.*

Exercise XII (a).

1. Find the circumference of a circle when the diameter in inches is (i) 2·9 (ii) 7·6 (iii) 13·8. ($\pi = 3{\cdot}142$).

2. Find the diameter of a circle when the circumference in cms. is (1) 152 (2) 98·7 (3) 256·4. ($\pi = 3{\cdot}142$),

3. If the mean diameter of the earth be 7912 miles, what is the circumference assuming (1) $\pi = 3\frac{1}{7}$ (2) $\pi = 3{\cdot}1416$?

4. A wheel 4 ft. in diameter made 750 revolutions in going through 2 miles. Find what distance is due to slipping of the wheels.

5. The diameter of a circular tank is 81·33 ft. What will it cost to enclose it with a fence at 1 anna per foot length ?

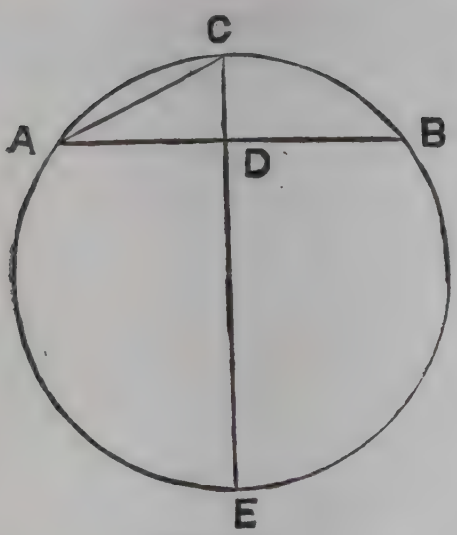


FIG. 114.

§ 95. Length of arc of a circle. ACB is an arc of a circle and C is the middle point of the arc, the length of the arc AB may be approximately obtained by the following formula (given by Hughens), *viz.*,

$$\text{arc} = \frac{8\text{AC} - \text{AB}}{3}$$

Example 1. If the chord AB is 8 inches and chord AC is 5 inches; find the arc AB.

Here arc $= \frac{8 \times 5 - 8}{3} = \frac{32}{3} = 10\frac{2}{3}$ inches.

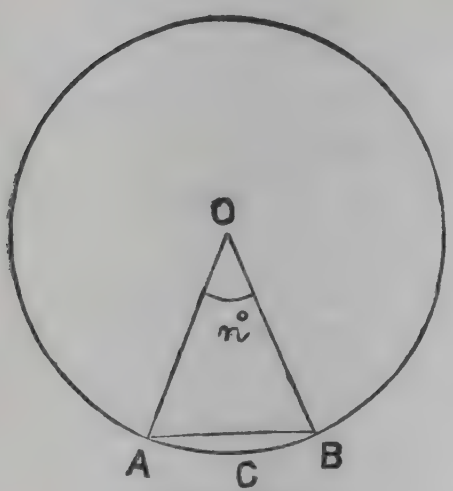


FIG. 115.

If ACB is a circular arch, AB is called the span of the arch and DC the heightof the arch.

Length of the arc of a circle subtending an angle of n° at the centre.

Because equal angles at the centre subtend equal arcs, we see that if angles are doubled, trebled, &c., the arcs are also doubled, trebled etc., *i.e.*, the arcs vary as the angles which they subtend at the centre.

$$\therefore \quad \frac{\text{arc AB}}{\odot^{ce}} = \frac{\angle \text{AOB}}{4 \text{ rt. angles}} = \frac{n^\circ}{360^\circ}.$$

$$\text{arc AB} = \frac{n^\circ}{360^\circ} \times 2\pi r = \frac{n}{180} \times \pi r.$$

Example 2. The circular arch of a bridge is 60 ft. long and the chord of half the arch is 28·4 ft. Find the length of the chord *i.e.*, the span.

$$\text{arc} = \frac{8AC - AB}{3}$$

$$\therefore \quad 60 = \frac{8 \times 28{\cdot}4 - AB}{3}$$

$$180 = 227{\cdot}2 - AB$$

$$\therefore \ AB = 47{\cdot}2.$$

i.e., the span is 47·2 ft.

Exercise XII (b).

1. In a circle of diameter 144 inches, find the length of an arc whose height is 16 in.

2. The chord of an arc is 24 ft. and the chord of half the arc is 5 yds. Find the length of the arc.

3. AB is a chord of a circle of radius 2 inches subtending an angle of 60° at the centre. Find the length of the minor arc using Hughens' formula, making also the required measurements. Check your answer by finding it in another way.

4. In a circle of 8 cm. radius a square is inscribed, find the length of the arc subtended by a side of the square.

§ 96. Area of a circle and of a sector.

A sector of a circle is the area enclosed by two radii and the arc intercepted between them.

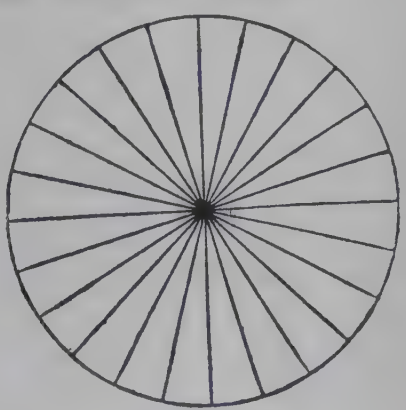


FIG. 116.

A circle may be divided into a large number of small

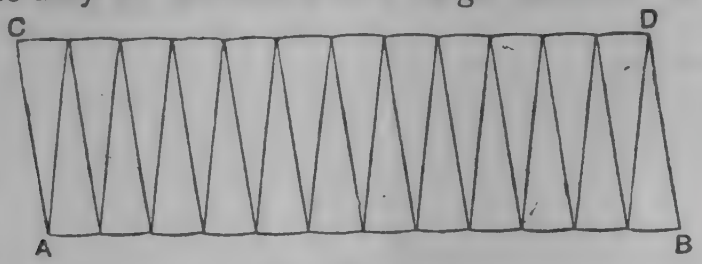


FIG. 117.

sectors as shown in Fig. 116 by dividing the circumference into a large number of small equal parts and joining the points of section to the centre. These sectors may be cut out and arranged on a line as in Fig. 117. The sectors in the upper half may be fitted into the corresponding recesses in the lower half. Thus the circle is equal to the parallelogram whose base is the semi-circumference and whose height is the radius [slant side may be taken approximately equal to the perpendicular height].

$\therefore$ area of a circle $= \pi r \times r = \pi r^2$.

Or thus:

It was proved in the chapter on mensuration of rectilineal figures that the area of a regular polygon = radius of the inscribed circle × semi-perimeter of the polygon. But the polygon tends to coincide with the circle as the number of sides is continuously increased.

The area of the circle is the limiting value of the area of the polygon when the number of sides is indefinitely increased.

$\therefore$ area of a circle $= r \times \pi r$ ($\because$ the perimeter of the polygon in the limit = the circumference of the circle).

Example 3. Find the area of the circle generated by the motion of the minute hand of a clock 6 in. in length.

$$\text{Area of a circle} = 3\tfrac{1}{7} \times r^2 = \frac{22}{7} \times 36 = \frac{11 \times 72}{7} = \frac{792}{7}$$
$$= 113\tfrac{1}{7} \text{ sq. in.}$$

Experimental Method.

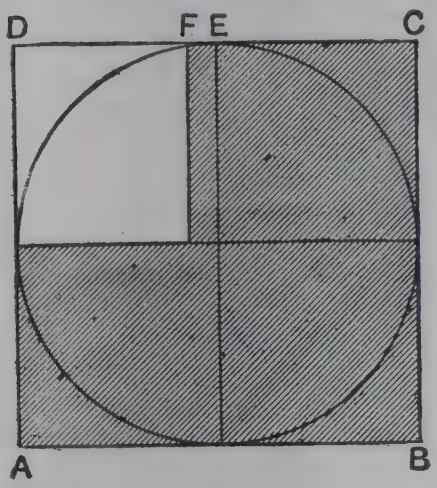


FIG. 118.

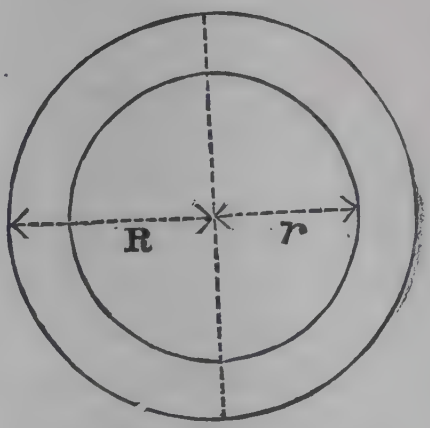


FIG. 119.

ABCD in fig. 118 is a square circumscribed to a circle and FE is taken equal to $\frac{1}{7}$ DE; so that the area of the shaded figure $= 3\frac{1}{7} r^2$ where r is the radius of the circle. In the figure r is taken as 1 inch. Cut a circle of radius 1 inch. out of cardboard, and also the shaded part of the square out of the same cardboard. Their weights will be found to be equal, hence the areas are equal.

An annulus. An annulus is the space between two concentric circles. (Fig. 119).

Area of an annulus = difference of the areas of the two circles.

$= \pi R^2 - \pi r^2$ where R and r are the radii of the outer and inner circles

$= \pi (R + r)(R - r)$.

Exercise XII (c).

1. Describe a circle of 2 in. radius on squared paper as in Fig. 120 and find the area by counting squares and thus verify the formula for the area of a circle. [To find the area of the circle, find the area of the quadrant shaded off and multiply by 4].

2. A circle is inscribed in a square and from it the circle is cut off. What is the area of the remaining portion?

3. A section of a round pillar is 5 ft. in diameter. Find the area of the section to the nearest square inch.

4. A donkey tethered by a rope can just graze over an acre of land. Find the length of rope to the nearest square foot.

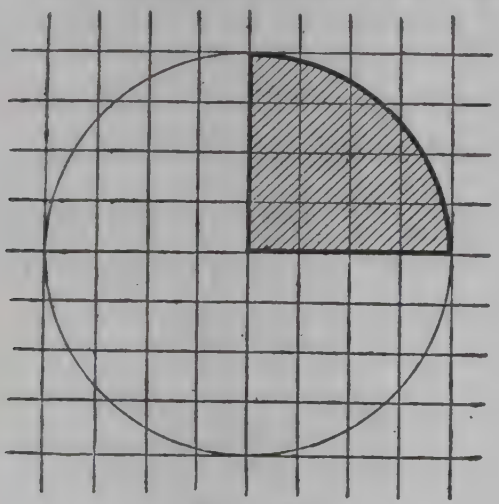


FIG. 120

5. A circular grass plot has a diameter of 70 yds. What area does it contain? Give your result in acres to two decimal places.

6. A circle encloses one quarter of an acre. Find the radius of the circle to the nearest yard.

7. A flower bed is in the form of an equilateral triangle (side 5 ft.) with semi-circles described upon the sides outside it. Find its area to the nearest square foot.

8. One circle is five times another in area. Compare their radii.

9. The diameter of a piece of copper wire is 0·168 inch. Find the area of the cross section.

10. A circular tower has an outer circumference of 17·25 metres and an inner circumference of 13·75 metres. Find the area covered by the wall.

Area of a sector.—As in the case of a circle, a sector may be divided into a large number of small sectors, and these small sectors arranged in a line as in Figs. 116 and 117, and we have

area of a sector = the radius × $\frac{1}{2}$ the length of arc of the sector.

Note.—Note the similarity in the formula for the area of a sector and that for a triangle.

Again became sectors of equal angles cut out of the same circle can be made to coincide, such sectors are equal. Therefore, when the angle of a sector is doubled,

trebled, &c., its area gets doubled, trebled, &c., *i.e.*, sectors of the same circle are to one another as their angles.

$\therefore \frac{\text{area of a sector}}{\text{area of a circle}} = \frac{n^\circ}{360^\circ}$ (where n° is the $\angle$ of the sector).

$\therefore$ area of a sector $= \frac{n}{360} \times \pi r^2$.

Example 4. A chord AB of a circle 8 ft. radius subtends an angle of 50° at the centre O. Find the area of the sector.

Area of the sector

$$= \tfrac{50}{360} \times \pi . 64 = \frac{80 \times 3{\cdot}1416}{9} = \frac{251{\cdot}328}{9}$$

$= 27{\cdot}925$ sq. ft.

§ **97. Area of a segment.**—Area of the segment ACB = area of the sector AOB — area of the $\triangle$ AOB [Fig. 115].

Exercise XII (d).

1. The bounding radii of the sector of a circle are 8 feet long and contain an angle of 60°. Find the area of the sector to the nearest sq. foot.

2. A chord of a circle of radius 3 feet subtends a right angle at the centre. Find the area of the smaller segment cut off by the chord to the nearest square inch.

3. A sector of a circle of 3 ft. radius is 125 sq. ft. in area. Find the length of the arc of the sector.

4. A chord of a circle is 6 in. long and equal to the radius of the circle. Find the areas of the two segments, it cuts off, in each case to the nearest square inch.

5. A sector of a circle has a perimeter of 12 feet and its arc subtends an angle of 50° at the centre. Find the radius of the arc to the nearest tenth of a foot.

6. A chord of a circle of radius 6 ft. subtends an angle of 60° at the centre. Find the area of the smaller segment cut off by the chord to the nearest square inch.

§ **98. Right prism and cylinder.** A right

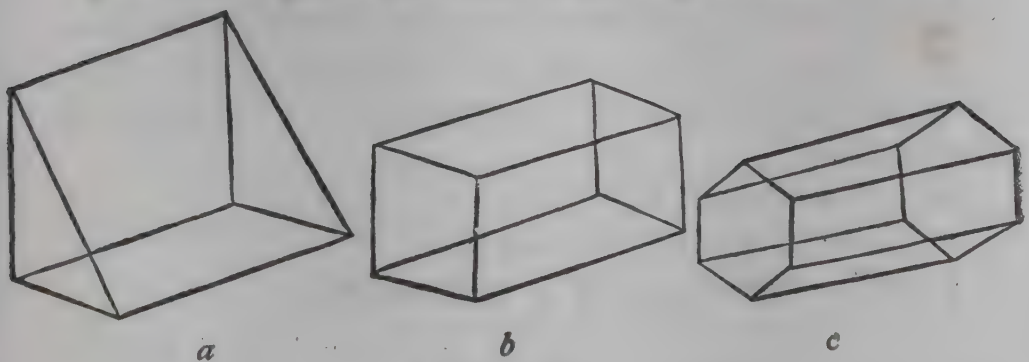


FIG. 121.

prism is a solid figure bounded by plane faces which are rectangles and two end planes which may be triangles, or polygons. The figures given here represent some of the different kinds of prisms. In fig. (*a*) the ends are equal △s and there are 3 faces which are rectangles. In fig. (*b*) the ends are equal squares. In fig. (*c*) the ends are hexagons.

When the ends are circles, you have what is called a **cylinder** see fig. (122). Just as the circle may be looked upon as the limit of a regular polygon when the number of sides is indefinitely increased, (see § 96), similarly a cylinder may be looked upon as the limit of a prism having for base a regular polygon when the number of sides of the polygon is indefinitely increased.

A round pillar and a round pencil are examples of a cylinder.

The cylinder may also be looked upon as a solid generated by the revolution of a rectangle about one of its sides kept fixed which is then spoken of as the *axis* of the cylinder.

(It may be interesting to note that the revolution of a point round an axis generates a line, the revolution of a line round an axis generates a surface, and the revolution of a plane surface round an axis generates a solid).

§ **99. Surface of prism.** The lateral suface= the area of all the rectangles forming the side boundaries $= ah + bh + ch + dh$..... where a, b, c, d,..... are sides of the polygonal base of the prism and h is the height of the prism.

$= h(a+b+c+d+\ldots\ldots)$

$= h \times$ the perimeter of the base.

§ 100. Surface of a cylinder.

Now if we take a prism whose base is a regular polygon and increase the number of sides of the polygon so that the length of each side becomes indefinitely small, the prism becomes a cylinder ; and since the above formula for the lateral surface of a prism is true, whatever the number of sides of the polygon be, we have

the curved surface of a cylinder $= h \times$ the perimeter of the base, h being the height of the cylinder.

In the limit the perimeter = the circumference of the ⊙ $= 2\pi r$ where r is the radius of the cylinder.

Hence the curved surface of the cylinder $= h \times 2\pi r = 2\pi rh$.

The curved surface of a cylinder may be got experimentally thus:

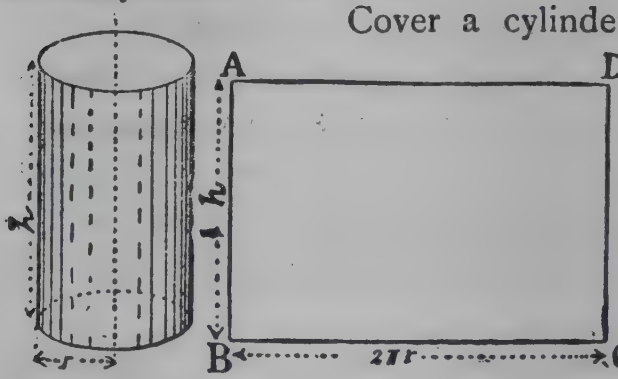


FIG. 122.

Cover a cylinder with a piece of thin paper, marking by a *, the extremities of the line where the edges come together. When the paper is unwrapped and laid on a plane, the paper will be found to be a rectangle whose sides are the height of the cylinder and the circumference of the base.

$\therefore$ Curved surface of a cylinder $= (2\pi r)h = 2\pi rh$.

The area of each of the circles which are the ends of the cylinder $= \pi r^2$.

$\therefore$ the total surface $= 2\pi rh + 2\pi r^2 = 2\pi r(h+r)$.

§ 101. Volume of a right prism.

Let ABC*abc* be a right prism whose edges A*a*, B*b*, C*c*, are at rt. angles to its ends ABC and *abc*. Let each of its ends contain S units of area and let h be its altitude. Take a plane $a'b'c'$ parallel to ABC and at unit distance from it. Then we see that the volume of the prism bounded by this plane and the end ABC consists of S unit cubes; for, on each unit area of ABC, there is a unit cube standing between it and $a'b'c'$.

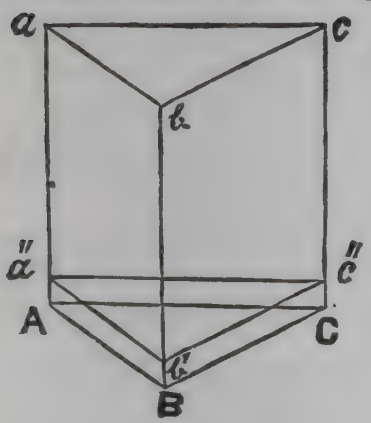


FIG. 123.

And since A*a* contains h units of length, the whole volume consists of h prisms each equal to ABC $a'b'c'$.

$\therefore$ the volume of the whole prism $= S \times h$.

Or in words, **The volume of a right prism is equal to the area of its base multiplied by its altitude.**

Note.—As the proof has no reference to the number of sides of the base, the proposition is true for a prism on any polygonal base.

§ 102. Volume of a cylinder.—The volume of a cylinder whose height is h and the radius of whose ends is $r = \pi r^2 h$.

For, the area of the base $= \pi r^2$; and the volume may be proved to be $\pi r^2 h$ as in the preceding proposition or may be derived from the volume of a prism by passing to the limit.

Volume of a hollow cylinder.

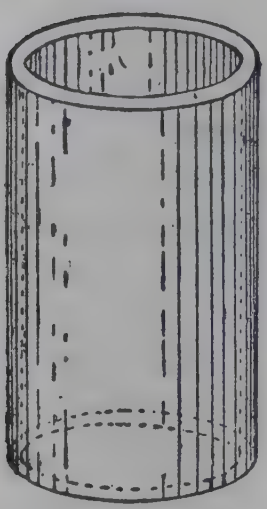


FIG. 124.

If R and r are the external and internal diameters, area of cross section $= \pi (R^2 - r^2)$ and hence volume of cylinder $= \pi (R^2 - r^2)h$.

Ex. 1. The radius of a cylinder is 3 inches. Height is 15 inches. What is (1) its lateral surface, (2) its total surface, (3) its volume?

1. Lateral surface $= 2\pi rh = 2 \times \frac{22}{7} \times 3 \times 15 = \frac{1980}{7} = 282\frac{6}{7}$ sq. in.

2. Total surface $= 2\pi r(r + h) = 2 \times \frac{22}{7} \times 3(3 + 15) = \frac{2376}{7} = 339\frac{3}{7}$ sq. in.

3. Volume $= \pi r^2 h = \frac{22}{7} \times 9 \times 15 = \frac{2970}{7} = 424\frac{2}{7}$ c. inches.

Ex. 2. Find (1) the lateral surface of a triangular right prism whose ends have edges 3, 4, 5 cm. and whose length is 1·5 dm. (2) its total surface (3) its volume.

(*a*) Lateral surface $=$ 1·5 dm. or 15 cm. the perimeter of an end $= 15(3 + 4 + 5) = 180$ sq. cm.

(*b*) Total surface $= 180 + 2 \times \frac{1}{2} \cdot 3 \cdot 4$ ($\because$ the $\triangle$ is rt.-angled) $= 180 + 12 = 192$ sq. cm.

(*c*) Volume $=$ area of the base $\times$ height $= \frac{1}{2} \times 3 \times 4 \times 15 = 90$ c. c.

Exercise XII (e).

($\pi = 3{\cdot}1416$ unless otherwise stated.)

1. A cylinder has a 5 cm. radius and its curved surface is 3630·408 sq. cm. What is its length? Find its volume.

2. The length of a right triangular prism is 6 cm. and its end has sides 2·5, 3·6, 4·7 cm. Find the area of its lateral surface and its volume.

3. What quantity of zinc is wanted to make a cylindrical tube 4 inches in diameter and 8 ft. long for carrying off the terrace water to the drain in a street?

4. A blacklead pencil (Faber's Taj Mahal) is 6·9 inches long; its diameter is ·3 inches. What amount of paper will be required to cover its curved surface? How many such pencils can be packed in a roll in the form of a hexagonal prism a side of an end being 1·5 inches and the height the same as that of a pencil?

5. Work out question (4) supposing the pencil is of the Kohinoor type whose cross section is a regular hexagon (side being ·3 cm. long and height 16·5 cm.)

6. A cylindrical gas-holder is 12 ft. high, the radius is 9 ft. How many cubic feet of gas will it hold? What quantity of sheet iron is required to make one such holder?

7. The base of a prism is in the form of a regular pentagon, each side being 2 feet; and the height is 15·4 feet. Find its volume.

8. The base of a prism has an area of 40 sq.cm. and the volume is equal to 2 litres. What is the height?

9. A well 4 metres in diameter contains 80000 litres. What is its depth?

10. The curved surface of a cylinder whose radius 4·3 in. and length 5·6 in. is unfolded into a rectangle and wrapped up to form a *new* cylinder. What are the dimensions of the new cylinder?

11. Find the volume of a right prism 8 ft. long whose ends are equilateral $\triangle$s of 14 inches side.

12. A cubic foot of brass is drawn into a wire 1500 yds. long, find the area of a section of the wire,

13. ABCD is the cross section of a right prism ·25 of a metre long. AB = 6 cm., BC = 18 cm., AD = 20 cm., and $\angle$A and $\angle$C are right-angles. Find the volume, and total surface.

14. Find the radius of a circle equal in area to the sum of two circles whose radii are 6 yds., and $17\frac{1}{2}$ yds.

15. Find the area of a plane circular annulus whose external and internal radii are 15·5 in. and 13·7 inches respectively.

16. The steam pressure on a piston 12 in. in diameter is 4250 lb. At what rate is this per square inch?

17. Find the cost of plastering the vertical surface of a column of brickwork 12 ft. high standing on an octagonal base whose side is 9 in. at the rate of 6 as. per sq. foot.

18. Find the weight of a right prism of silver 4 in. long the ends being triangles whose sides are 1·6 in., 1·8 in., 1·3 in. [1 cub. in. of silver weighs 6·1 oz. nearly.]

19. A level heading 12 ft. wide at the botton, 9 ft. wide at the top and 8 ft. high is driven into a seam of coal of which 40 cub. ft. go to the ton. How many tons of coal have been obtained when the length of the heading is 120 yds.?

20. A trench is 16 ft. at the top and 12 ft. at the bottom having a uniform depth of 6 ft. If its length is 50 ft., how many gallons of water will it hold?

21. How many square yards are covered in 25 revolutions of a cylindrical roller whose length is 5 ft. and diameter 4 ft.?

22. Find the surface of a hollow cylinder whose external diameter is 40 in., thickness $1\frac{1}{2}$ in. and height 3 ft.

23. Find, to the nearest pound, the weight of 20 ft. of lead piping whose internal diameter is $1\frac{1}{2}$ in. and thickness $\frac{1}{8}$ in., given 1 cub. in. of lead weighs $6\frac{1}{4}$ oz.

24. Show that a cube is a particular kind of prism.

The edge of a cube is 4 in. and its weight is 30 lbs. What will be the weight of a cube of the same material, the edge of which is $2\frac{1}{2}$ inches?

25. The base of a prism is a square of 4 ft. side, and its height is 8 ft. The base of another prism is a square of 6 ft. side and its height is 19 ft. Compare the volumes and the surfaces of the two prisms.

26. The dimensions of the different scantlings in a truss are as follow :—

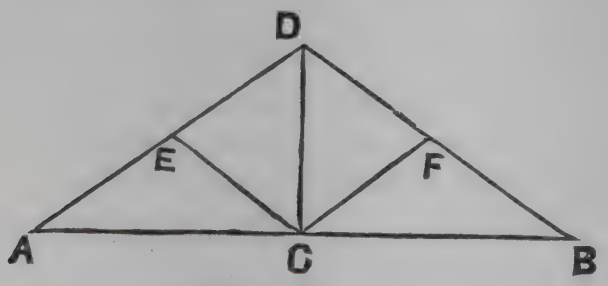


FIG. 125.

AB=21 ft. × 8 in. × 5 in.
CD=8 ft. × 5 in. × 4 in.
AD and BD each = 13 ft. × 6 in. × 4 in.
CE and CF each = 6½ ft. × 5 in. × 4 in.

How many cubic feet of timber will the truss contain ?

27. This figure (not drawn to scale) shows a kind of truss used

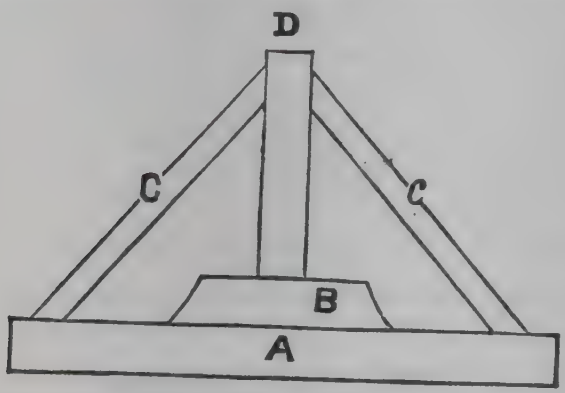


FIG. 126.

in Indian roofing. The dimensions of the scantlings purchased are as follow :—

A 12 ft. × 7 in. × 5 in.

B Length at top = 9 in., at bottom = 10 in. } Height = 6 in.
Breadth = 7 in.

C.....each a prism on a square base = 8 ft. × 4 in. × 4 in.

D.....a prism on a square base = 5 ft. × 4 in. × 4 in.

How many cubic feet of timber will the truss contain ?

28. A cyclindrical gasometer with a flat top is required to hold a million cubic feet of gas. If the diameter is 150 ft., find the height. Determine its weight assuming that the average weight of the iron plate and framing to be 8 lbs. per sq. ft. of surface.

29. An oil drum is in the form of a cylinder. It is 12·5 ft. long and its diameter is 6·3 ft. Find how many gallons of oil it can hold, given that 1 gallon = 244·271 cub. inches.

30. A boiler contains 80 copper tubes each $\frac{7}{8}$ in. outside diameter d $\frac{1}{4}$ in. thickness and 6 ft. long. Find the total weight of the tubes.

31. A cylinder 9 ft. diameter and 10 ft. long contains 34,400 lbs. of petroleum. Find the specific gravity of petroleum. [Given one cubic foot of water weighs 1000 oz.]

32. A piece of metal (sp. gr. 10) 3 ft. long, 5 in. broad and $\frac{1}{4}$ in. thick is drawn out into wire of uniform diameter $\frac{1}{12}$ in. Find the length in yards and the weight of the wire.

33. The external diameter of an iron pipe is 10 in. and the internal diameter 9·25 in. Find the weight of a length of 9 ft. of this piping, weight of 1 cub. inch = ·28 lbs.

34. A steel plate is $\frac{1}{2}$ in. thick with two circular pieces 1 ft.

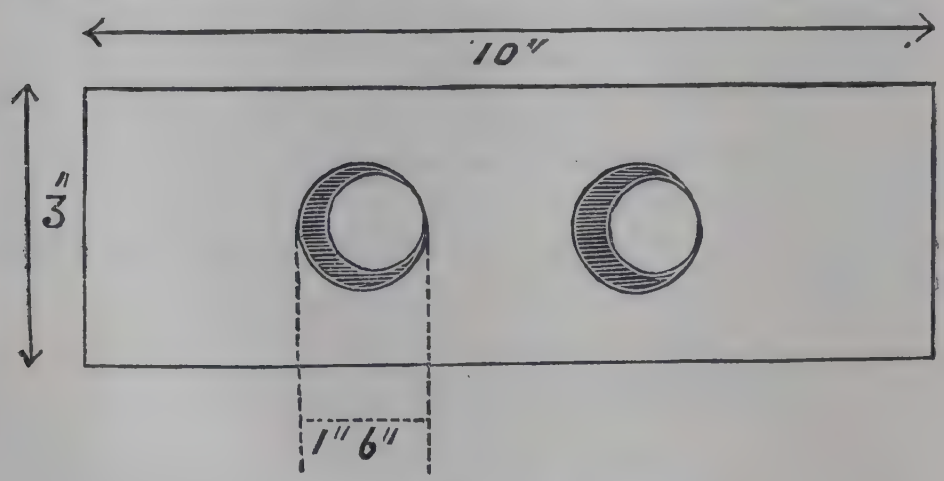


FIG. 127.

6 in. diameter cut from it and is rectangular in form. Find its weight, weight of 1 cub. in. = ·29 lbs.

35. A wrought iron plate is in the form of an equilateral △ each side being 4 ft. 6 in., and is $\frac{3}{4}$ in. thick. There is a hole 9 in. in diameter cut through it as in figure 128. Find the volume. Weight of 1 cub. in. = ·28 lbs.

36. The dimensions of a metal bar are 8 in. by 6 in. It is

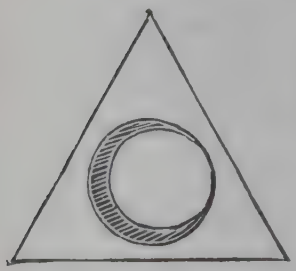


FIG. 128.

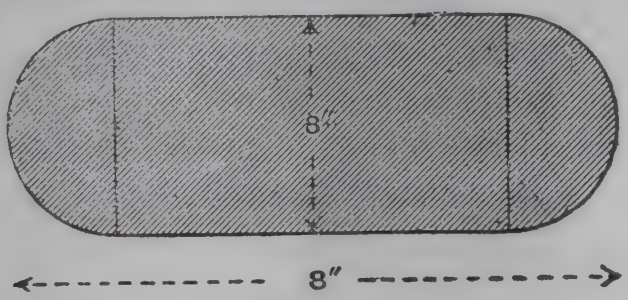


FIG. 129.

rounded off in the form of a semicircle at each side. Find the weight of 30 ft. length. S. G. of the metal = 6·3.

37. Post office weights are 3 in. diameter and contain a slot ·4 in.

FIG. 130.

wide and $1\frac{3}{4}$ in. long as in Fig. 130. Find in each the thickness when the weights are $\frac{1}{2}$ lb. and 2 lbs., given 1 cub. in. of iron weighs ·27 of a lb.

CHAPTER XIII.

DISCOUNT AND EXCHANGE.

§ **103. Discount.** The term **Discount** is generally applied to the reduction made in a "bill" in consideration of an immediate or cash payment.

A retail shopkeeper often allows a customer who pays cash a certain reduction of, say 10 %, from his bill or account. This reduction is known as *Cash Discount*. Thus, if a customer buys goods to the value of Rs. 35 and is allowed 10 % cash discount, he actually pays only (Rs. 35—Rs. $3\frac{1}{2}$) or Rs. $31\frac{1}{2}$.

Again, suppose A owes B a sum of Rs. 105, payment being due at the end of a year and that B wishes to be paid immediately; then it is clear that a fair arrangement would be, the use of money being valued at 5 %, that A should pay B as much as would amount to Rs. 105 in a year at 5 %, *i.e.*, A should pay Rs. 100. This reduction of Rs. 5 from the amount due, is called the **Theoretical** *or* **true discount** and the money that is to be paid, *viz.*, Rs. 100 is called the **Present value** of the amount due, *viz.*, Rs. 105.

But in practice, interest is charged on Rs. 105 at 5 % for 1 year and is deducted from Rs. 105. The discount in this case = Rs. $\frac{5}{100} \times 105$ = Rs. $5\frac{1}{4}$ and the amount paid is Rs. 105 — Rs. $5\frac{1}{4}$ = Rs. $99\frac{3}{4}$. Rs. $5\frac{1}{4}$ is called the **Commercial** or **Bankers' discount.**

The *true discount* is not in use in practical business, consequently we shall not hereafter deal with problems on

true discount. And by the term 'Discount' we shall always mean *Bankers' discount* unless otherwise specified.

Ex. Find the discount on a sum of Rs. 500 due after 6 months at 5 per cent.

Discount $= \text{Rs. } 500 \times {}_{1}\frac{5}{00} \times \frac{1}{2} = \text{Rs. } 12\frac{1}{2}$.

§ **104. Bill of Exchange,** Suppose a person A. B. buys goods of a merchant P. Q. for which cash payment is not made. The following is one of the courses adopted. P. Q. draws a bill, called a bill of exchange, as shown below, and sends it for the acceptance of A. B.

INLAND BILL OF EXCHANGE.

Rs. 405.	*Accepted, payable at the Bank of Madras.* A. B.	MADRAS, *9th January*, 1910.
Three months me or my order Five Rupees for		after date pay to Four hundred and value received.
To Mr. A. B. *Calicut.*		P. Q.

On receipt of this bill, A. B. signs his name across the bill as having accepted it and sends it back to P. Q.

Now P. Q. presents the bill at the expiry of 3 months to A. B.'s banker, the Bank of Madras, and receives Rs. 405.

If P. Q., or the person authorised by him to receive the Rs. 405, requires the money before the due date, he may present the bill at any bank, where bill-discounting is part of the business, and get the cash equivalent of the sum mentioned in the bill. The banker deducts by way of

discount the interest on Rs. 405 for the period that the bill has yet to run. This discount, which is nothing but the simple interest on the amount of the bill, is called the **bank** or **commercial discount,** as distinguished from the **true** or **theoretical discount** which is the interest on the *present value* of the bill amount.

A bill of exchange, bill or **draft** is thus a letter of request addressed by a person who is called the **drawer** to another person, called the **drawee,** desiring him to pay a certain sum of money either to the drawer himself or to a third party called the **payee** within a specified time.

Bills of Exchange may differ as to the periods for which they are to run. Some bills are drawn payable at sight, and are called *sight bills* or *sight drafts.*

They may again be payable at a certain number of days after sight and may have some time to run after being drawn up Such bills are called *time bills* or *time drafts.*

The *face value* of a bill is the amount shown in the paper. *e.g.*, Rs. 405, in the above illustration.

A promissory note or note of hand is a document in which a person promises to pay another a sum of money at some specified time.

FORM OF A PROMISSORY NOTE.

Rs. 200.

MADRAS, 4*th March*, 1913.

Six months after date, I promise to pay T. K. Swaminatha Aiyar or order Two hundred Rupees, value received.

N. ARUNACHELLA AIYAR.

Thus, a promissory note is, in fact, a bill in which the drawer and the acceptor are identical.

Promissory notes may be discounted quite like bills of exchange, and a banker is said to discount a bill or note of hand, when he buys it of the holder for the amount mentioned in the bill or note, diminished by the mercantile discount on it.

A *bank note* is a promissory note issued by a banker payable to bearer on demand, and *bank cheque* is a bill of exchange drawn on a bank.

§ **105. Days of Grace.** Payment for Inland Bills of Exchange cannot be legally demanded until 3 days called "Days of Grace" after the day on which it is nominally due. "If the due date should fall upon a Sunday, Christmas Day, Good Friday or day appointed by Royal proclamation as a public fast or thanksgiving, the bill is due on the preceding business day; should it however fall due upon a Bank Holiday (other than Christmas Day and Good Friday) or should the first day of grace be a Sunday and the second day of grace a Bank Holiday, the bill is payable on the succeeding business day. *A month* always means *a calendar month.* Thus a bill drawn on May 3rd at one month after date, is due on June 6th and bill drawn on August 30th and August 31st at one month after date are each due on October 3rd." Days of grace are always allowed unless the contrary is stated.

Ex. Calculate the Banker's discount on a bill for Rs. 1,095 drawn April 1st at 4 months and discounted on July 5th at 5 %

The bill is nominally due on the 1st of August and legally due on the 4th of August. It is discounted on the 5th of July, *i.e.*, 30 days before the due date. The interest on Rs. 1095 for 30 days at 5%.

$= \text{Rs. } 1095 \times \frac{30}{365} \times \frac{5}{100} = \text{Rs. } 4\frac{1}{2}.$

Exercise XIII (a).

1. What sum should you receive for a note of Rs. 400 due on 15th Nov. and discounted on the 5th of Oct. at 5 per cent?

2. Find the banker's discount on £292 drawn on the 5th of Nov. at 1 month and discounted on the 13th of Nov. at 5 per cent.

3. Find the banker's discount on a bill for £219 drawn on the 4th of March at 3 months and discounted on the 8th of April at 8½ per cent.

Calculate the discount on the following bills :—

4. Rs. 683-10-8 dated Aug. 7th at 2 months discounted Sep. 1st at $2\frac{5}{8}\%$.

5. Rs. 2,850 drawn Oct. 1 at 6 months and discounted Jan. 4th at $4\frac{1}{2}\%$.

6. Rs. 3,450 maturing on March 12th and discounted on Feb. 3rd at $5\frac{1}{2}\%$.

7. Rs. 9,540 nominally due Oct. 1st and discounted May 29th at $4\frac{1}{16}\%$.

§ **106. Exchange.** In actual practice, in payments on foreign bills of exchange, not only has the discount for payments before the due date to be calculated, but also the equivalent of the sum of money has to be found in the coin of the country of payment.

The following table gives the approximate values of the standard coins of some of the chief countries in terms of British money and the number of corresponding coins equivalent in intrinsic value to £1 sterling. The approximate values of these standard coins in terms of Indian money are found by calculating at Rs. 15 to the £.

Country.	Standard Coin (gold).	Approximate value in English Money.	Approximate value of £1 sterling.
France ...	Franc = 100 centimes ...	$9\frac{1}{2}d.$	25·22 francs, lira, etc.
Belgium ...			
Switzerland ...			
Italy ...	Lira = 100 centesimi.		
Greece ...	Drachma = 100 Septa ...		
Spain ...	Peseta = 100 centesimos ...		
Germany ...	Mark = 100 Pfennig ...	$11\frac{3}{4}d.$	20·43
Austria ...	Florins = 100 kreutzers ...	1*s.* $11\frac{3}{4}d.$	10·1
	Krone = 100 Hellers ...	10*d.*	24
Russia ...	Rouble = 100 copecks ...	3*s.* 2*d.*	6·3
Norway ...	Krone = 100 Ore.	1*s.* $1\frac{1}{4}d.$	18·2
Sweden ...			
Denmark ...			
Holland ...	Florin = 100 cents = 20 Stivers	1*s.* 8*d.*	12·1
Portugal ...	Milreis = 1000 Reis	4*s.* $5\frac{1}{4}d.$	4·5
United States of America ...	Dollar = 100 cents.	4*s.* $1\frac{1}{4}d.$	4·87
India ...	1 sovereign = 15 rupees ...	£1	1

The following account of bills of exchange will give an idea as to how accounts are adjusted between two countries supplying goods to each other without the actual transmission of money and how a necessity arises for calculating discount.

The payment of a sum of money due to a merchant in a foreign country may be made by sending:

(i) SPECIE, that is, coins;

(ii) BULLION, that is, gold or silver in bars; or

(iii) A BILL OF EXCHANGE.

(i) If a person in London has a payment of 500 Napoleons to make in Paris, and if he decides upon making this payment by sending *specie*, it will be necessary for him first to ascertain the value of a pound sterling in French money. This value depends on the weights and fineness of the metals composing the *standard* coins of the two countries.

The established value of the standard coin of one country expressed in terms of the standard coin of another, is known as the par of exchange.

The **Par of Exchange** is that sum in the currency of one country which in *intrinsic* or real value is equal to a fixed sum in the currency of another country. Thus, according to the Mint Regulations of England and France, £1 sterling = 25·22 francs, and the exchange is said to be *at par* when made at this rate.

NOTE.—Par Value is liable to variation, since the values of different metals are not always in the same ratio.

(ii) Payment in bullions is much the same as payment in specie, the weights and fineness of the bullions as also the ratio of the prices of the metals having to be noted.

(iii) The necessity of payment in specie or bullion is generally obviated by the method of payment by bills of exchange. Let us take two merchants A and B in Madras, and two merchants C and D in London. And let us suppose that A owes C £500, and D owes B the same sum. Instead of A sending cash to C, and D to B, C may ask A to pay £500 to B and B may ask D to pay C £500; thus C and B get paid the amounts due from A

and D respectively. Thus the transaction is settled without any money being sent from Madras to London or from London to Madras.

In actual practice, it would be difficult for C to find a person D who owed to Madras exactly as much as he ought to get from Madras. This difficulty can be surmounted if the business is done through a bank having branches both in London and Madras. In this case C would draw a bill on A for £500 and A would accept it and B would draw a bill on D for £500 and D would accept it. A might pay the bill into the London Branch and the Bank would receive the money in Madras. D would pay the bill into the Madras Branch and the bank would receive the money in London.

The account of the Bank would then stand thus :—

London Branch.		Madras Branch.
Has to pay C £500 for a bill on A for £500.	→	Gets from A £500 for his bill.
Gets from D £500 in exchange for his bill.	←	Has to pay to B £500 for a bill on D for £500.

The bank thus transfers no coins but has only to send bills from one branch to the other. These amounts need not be equal. For instance, say on a certain day, into the London Branch are paid 200 bills on Madras (payable in Madras) amounting to £10,500 and on the same day 150 bills on London (payable in London) are paid into the Madras Branch amounting to £9,500. The bank would receive £10,500 in Madras due to London and £9500 in London due to Madras. ∴ the Bank would have to remit £1000 to London to balance the account. The £1000 need not be sent; the Madras Branch may

be written down as debtor to the London Branch. On another day the balance might be against London. In any case there need be no sending of coin until the debt is very large.

Note. As in every branch of business, we have brokers dealing in bills buying and selling them. Their charge for buying bills is usually 1 per mille—*i.e.*, $\frac{1}{10}$ per cent.

[Compare the system of payment by "Hundi" in Southern India. The Hundi is only a Bill of Exchange. A merchant in Tuticorin, to whom money is owed by a merchant at Madras, may direct the payment to be made to his own agent at Madras or to a third merchant at Madras to whom he owes money. Accounts are thus adjusted.]

Foreign Bills of Exchange are bills drawn in one country and made payable in another; when they are drawn and payable in the same country they are called INLAND BILLS.

[Inland Bills of Exchange have been fully explained already. (§ 104)].

'Short exchange or exchange at short' refers to bills drawn at sight, or at short sight, which is usually three days.

§ 107. Course of exchange.—The market price of bills is not always determined by the Par Value of the standard coins, but is primarily regulated by the state of trade between countries concerned. If the value of the exports to any country is equal to the value of the imports from the same country, the trade is said to be *balanced*. In this case the bills drawn in each country upon the other

On the reverse is given a sample form of a Foreign Bill of Exchange. These bills are generally drawn up in sets of three. The sample given is the first, and it is so marked; hence also the reference on it :— "Second and Third unpaid." This is to prevent any inconvenience in case the first be lost or delayed in transmission.

Drawn under Letter of Credit No. ..... dated ..... for £ .....

No. [redacted]

Exchange for £ [redacted] 19..

At ..... after sight pay this FIRST of Exchange.

(Second and Third unpaid) to the order of .....

[redacted]

value received and charge the same to account .....

To .....

.....

will be equal in amount, and the rate of exchange will be at par. But if there is more money payable at the moment by people, say, in India to people in England, than there is payable by those in England to those in India, there will be a demand in India for orders on England and a premium will be payable for the accommodation by those who want them. In this case, the *exchange in India is said to be* **above par** *or* **at premium,*** *being* **against India** *and* **in favour of England**. In England, on the contrary, there will be more people ready to give such drafts than there are in want of them, and those who dispose of them must do so at a discount. And the exchange in England is said to be **below par** or **at discount.**

The student will now understand the following definition of 'Course of Exchange.'

---

* It will be well to point out that, though exchange may be at a premium, this premium can, in no case, rise above the cost of remitting the money in specie. An English merchant who has to remit money to Paris, finds that the rate of exchange between London and France is £1=25·22 francs and the cost of remittance is 10 cents per £; so he has only 25·12 francs left to pay his debt. But if bills on Paris are sold at a rate of 25·15 francs per £ it is more advantageous to buy bills on Paris than to transmit money by coins. It is clear that if bills on Paris are sold at a lower rate than 25·12 francs per £ then nobody will buy these bills. Hence 25·12 is **called a specie point.** Similarly a merchant in Paris to remit money to England will have to pay 25·22 francs +10 cents, *i.e.*, 25·32 francs for each £ due in London. If bills on London are sold at 25·28 francs per £, he will rather buy bills on London than remit coins to England. But if bills on London are sold at a higher rate than 25·32 francs per £ it is clear that nobody in Paris will buy these bills. This is another specie point. Thus the *course* or rate of exchange between England and France may vary between the specie points, *i.e.*, 25·12 and 25·32 francs per £.

The **Course of Exchange** at any date *means* that *variable* sum in the currency of one country which on that date is reckoned as equivalent to a *fixed* sum in the currency of another country.

§ **108.** It may sometimes be advantageous not to remit money directly to a foreign country, but through one or more other foreign countries Thus, if a London merchant has a remittance to make to Paris, he may obtain a bill upon Antwerp, and may have his money remitted there; from Antwerp to Paris by another bill drawn upon Paris at Antwerp. This calculation of the course of exchange between two places, as determined from a comparison of the courses of exchange between those two places and one or more intervening places, is known as **Arbitration of Exchange.**

The Arbitration is said to be **simple** when only one place intervenes, and **compound** when more places than one intervene. The rate determined by such arbitration is known as the **arbitrated rate of exchange.**

In conclusion: If a merchant A in England sells goods to B in India at Madras, the debt of B to A may be discharged in either of the two following ways:—

(1) A may draw a bill on B (Madras) which the latter accepts. A then sells the bill to a Bill-broker or pays it into his bank. The bill may be transferred by the bank to its branch in Madras or to some other bank doing business with the former, or be presented to B and paid.

(2) B may buy a bill or a draft on London (that is payable in London) and send it to A who then obtains cash for it.

The following examples illustrate two of the most common methods stated above, employed by a person in India (at Madras) in liquidating a debt abroad, say at London :—

In Newspaper extracts the course of exchange is given thus : New York $48\frac{11}{16}$ to $48\frac{15}{16}d$. When 2 prices are given it means that the sellers of the bills get the first and buyers pay the second.

Ex. 1. Find the cost of settling a debt of £1,000 by buying from a bill-broker 3 months' bills on London at 1*s.* $4\frac{1}{8}d$. per rupee, discount $2\frac{1}{2}\%$ brokerage and stamp duty 1·5 per mille.

Method 1.

Debt = £1,000 if paid now

Interest for three months $= 1000 \times \frac{5}{2 \times 100} \times \frac{1}{4} = £6\frac{1}{4}$

Thus £1006$\frac{1}{4}$ = the money to be paid at the end of three months at the end of which time the bills mature. In addition, there is the brokerage and stamp duty at 1·5 per 1,000.

$\therefore$ the total amount = £1007$\frac{3}{4}$.

$\therefore$ amount to be paid to settle the debt in rupees

$$= \text{Rs. } \frac{4031}{4} \times \frac{20}{1\frac{11}{32}} = \frac{4031}{4} \times \frac{\overset{5}{\cancel{20}} \times 32}{43} = \frac{4031 \times 160}{43}$$

= Rs 14999-1-1 to the nearest pie.

Ex. 2. Find the cost of settling a debt in London of £1000 by employing an agent in London to draw a three months' bill on Madras at 1*s* $3\frac{3}{4}d$. per rupee, rate of discount, brokerage and stamp duty being the same as in Ex. 1.

The agent draws a three months' bill or draft for the debtor at Madras to accept. The rate of exchange is 1*s.* $3\frac{3}{4}d.$ per rupee, *i.e.*, 1*s.* $3\frac{3}{4}d.$ paid now in London will buy a bill for 1 rupee payable after 3 months at Madras. By spending 1 rupee in 3 months' time the debtor can liquidate a present debt of $15{\cdot}75d.$ But brokerage and stamp duties amounting to 1·5 per 1000 have to be paid, which come to $\cdot 023625d.$ on $15{\cdot}75d.$

Re. 1 expended in 3 months' time liquidates a present debt of $15{\cdot}75 - {\cdot}023625$ or $15\,726375d.$ in London. But one rupee expended in 3 months' time = a present payment of Re. $(1 - \frac{1}{400} \times \frac{1}{4}) =$ Re. $\frac{159}{160}$.

A debt of $15{\cdot}726375d.$ is settled by a present payment of Re. $\frac{159}{160}$.

$$\therefore \text{ debt of } £1000 \text{ by Rs. } \frac{159}{\cancel{160}_{2}} \times \frac{1000 \times \overset{3}{\cancel{400}}}{15{\cdot}726375} = \frac{477000}{31{\cdot}45275}$$

$$= \text{Rs. } 15165{\cdot}60$$

$$= \text{Rs. } 15166 \text{ correct to a Re.}$$

Ex. 3. If the price in Paris of 3 months' bills on London be 25·08 francs per 1 £ and the rate of discount in London is 3 per cent., to what "short" rate is this equivalent.

25·08 francs paid in Paris will buy a bill on London for £1 payable at the end of 3 months.

Now present value of this £1 $= 1 - \frac{1}{4} \times \frac{3}{100} \times 1 = \frac{397}{400}$.

$\therefore$ £$\frac{397}{400}$ is equivalent to 25 08 francs

$\therefore$ £1 is equivalent to $\frac{25{\cdot}08 \times 400}{397} = \frac{10032}{397}$

$= 25{\cdot}27$ francs nearly.

Ex. 4. Find the arbitrated rate of exchange between London and Vienna in florins for £1, when the rate of exchange between London and Petersburg is $31\frac{7}{8}d.$ for 1 rouble, and that between Vienna and Petersburg is $95\frac{5}{8}$ florins for 60 roubles.

No. of florins required = £1

£1 = 240*d.*

$31\frac{7}{8}d.$ = 1 rouble.

60 roubles = $95\frac{5}{8}$ florins.

$\therefore$ no. of florins required $= \frac{240 \times 95\frac{5}{8}}{31\frac{7}{8} \times 60} = 12.$

N.B. This process is called *chain rule*, the right-hand side of one step and the left-hand of the next step being of the same denomination. Hence the link is continuous.

Ex. 5. On a certain day the exchange in London for 3 months' bills on Madras was 1*s.* $3\frac{3}{4}d.$ per rupee and the exchange in Madras on London was 1*s.* $4\frac{1}{8}d.$ per rupee. A Madras merchant had to pay a debt of £9030 Assuming the rates of discount in London and Madras to be respectively 3 °/₀ and 5°/₀, should he remit bills to London or instruct his agent in London to draw on Madras and what is the cost to him by each method?

By the first method, 1 rupee paid now will buy a bill for 1*s.* $3\frac{3}{4}d.$ due 3 months hence in London

$\therefore$ 1 rupee $= \frac{21}{16}s. - \frac{1}{10} \times \frac{3}{100} \times \frac{1}{4} = \frac{21}{16} \times \frac{397}{400}s.$

$\therefore$ £9030 or $9030 \times 20s. =$ Rs. $\frac{16 \times 400}{21 \times 397} \times \overset{430}{\cancel{9030}} \times 20$

$= \text{Rs.}\frac{55040000}{397} = \text{Rs. } 138640$ correct to a Re.

According to the second method 1*s.* $4\frac{1}{8}d.$ paid now will buy a bill for Re. 1 payable 3 months hence at Madras.

$$\therefore \frac{43}{32}s. = 1 - \frac{\cancel{8}}{\cancel{100}\atop 20} \times \frac{1}{4} = \text{Rs. } \frac{79}{80}.$$

$$\therefore \quad 9030 \times 20s. = \frac{79}{\cancel{80}\atop \cancel{8}} \times \frac{\overset{210}{\cancel{90}}3\cancel{0} \times \overset{4}{\cancel{10}}\cancel{0} \times \overset{2}{\cancel{3}}\cancel{1}}{\cancel{43}} = \text{Rs. } 132720.$$

The second is preferable.

Exercise XIII (b).

1. Find the cost to a Madras merchant of settling debts of (1) £250, (2) £375, (3) £425, assuming the following rates of exchange for 3 months' bills :—

(i) Bombay on London 1*s.* $4\frac{1}{16}d.$ per rupee.

(ii) Bombay on London 1*s.* $4\frac{3}{8}d.$ per rupee.

(iii) London on Bombay 1*s.* $3\frac{9}{16}d.$ per rupee.

(iv) London on Bombay 1*s.* $3\frac{7}{8}d.$ per rupee.

Rate of discount $4\frac{0}{00}$. Brokerage 1 per mlle. Stamp duty 0·5 per mlle.

2. Find the cost to a London merchant of settling a debt of 650,000 dollars in New York assuming that the rates of exchange for 3 months' bills sare as follows :—

(i) London on New York 4·95.

(ii) New York on London 4·75.

Rate of discount and stamp duties as in 1.

3. Suppose the rate between London and Hamburg is 20·55 ; Hamburg and Vienna is 180 marks per 100 florins ; Vienna and Paris is 48 florins per 100 francs. Find the cost of transmitting 2500 francs from London and Paris through Hamburg and Vienna. Work by the chain rule.

4. The par of exchange between England and India is 1*s.* 4*d.* per rupee; a money-lender gives 14 rupees for £1 and 1 rupee for 1*s.* 2½*d.* A traveller changes Rs. 3360 into pounds and on his return 15 pounds into Indian money. Find in rupees the whole profit made by the money-changer on both transactions.

5. A merchant in New York wishes to remit to Calcutta 5060 dollars, a dollar being equal to 4*s.* 6*d.* and 1*s.* 4*d.* = 1 rupee. For what sum must he draw his bill when bills on India are a premium of 10 per cent.?

6. If the exchange in London of 3 months' bills on Paris be 25·48 and discount there 4½ %, to what short rate is this equivalent and what debt in Paris can be discharged by a London merchant who holds a three months' bill on Paris for £600.?

7. A person in London owes another in Japan 10.000 yens which must be remitted through St. Petersburg. He pays the requisite sum to his broker when the exchanges between England and Russia are £1 = 6·3 roubles and 1 rouble = 1·55 yens. The remittances are delayed until the exchanges have altered to £1 = 6·4 roubles and 1 rouble = 1·65 yens. What does the broker gain or lose by the delay.?

8. A banker has orders to remit bills on Berlin at 20·65 marks and to draw on St. Petersburg at 24½*d.* per rouble. If the Berlin quotation has fallen to 20·62, at what quotation may he still draw on St. Petersburg so that the rate of exchange between St. Petersburg and Berlin may remain unaltered.?

CHAPTER XIV.

PYRAMID, CONE, SPHERE.

§ **109. Pyramid.** A pyramid is a solid figure bounded by triangular plane faces and a base which may be any rectilineal figure. All faces except one meet in a point called the vertex. The base may be a triangle in which case the pyramid is called a triangular pyramid or a tetrahedron. In Fig. 131 ABCD is the base. P is the vertex and the triangular plane faces are PAB, PBC, PCD and PAD.

In Fig. 132 PABC is a tetrahedron.

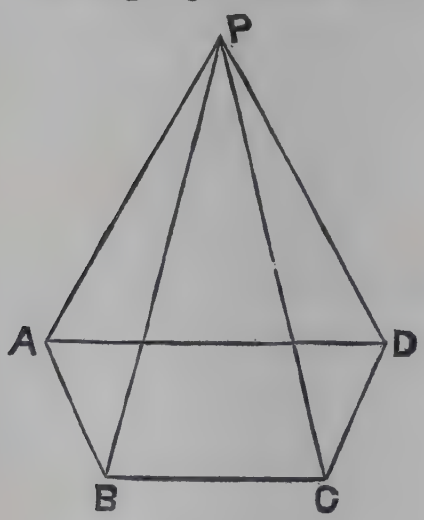


FIG. 131.

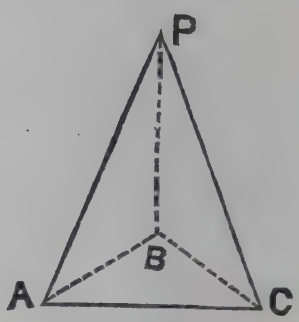


FIG. 132.

Surface of a pyramid = the area of the base and sum of the areas of the triangular faces.

If the base is regular, and the faces are all equal,

the *lateral surface* = $n \times$ the area of a triangle of which the base is a side of the polygon [where n is the number of sides]

$= n \times \frac{1}{2}$ a side of the polygon $\times$ slant height (which is the $\perp$ drawn from the vertex of the pyramid on a side)

$= \frac{1}{2}$ the perimeter of the polygon $\times$ slant height.

The total surface = lateral surface + area of the base.

§ 110. Equivalence of pyramids or prisms on the same base and of the same altitude. Place a pack of cards on a table so that the whole is in the form of a cuboid. The cards may be slipped so that each overlaps the one below it to the same extent. The pack now takes the form of an oblique parallelopiped. Without altering the height of the pack, the number of cards may be made infinitely great and consequently the thickness and overlap of each infinitely small. The solid thus formed would then really be an oblique parallelopiped and the volume would remain unaltered. Thus **parallelopipeds having the same base and height are equal.** Each has the same volume as the cuboid of the same base and height.

By a similar reasoning it may be proved that **pyramids standing on the same base and having equal altitudes are equal.**

§ 111. A right triangular prism can be divided into three equal triangular pyramids.

The right prism ABCFDE (Fig. 133) consists of the pyramids ADEF, ABCF and ABEF.

The pyramid

ADEF has the base DEF and altitude AD.

ABCF has the base ABC which is equal to DEF and altitude CF which is equal to AD.

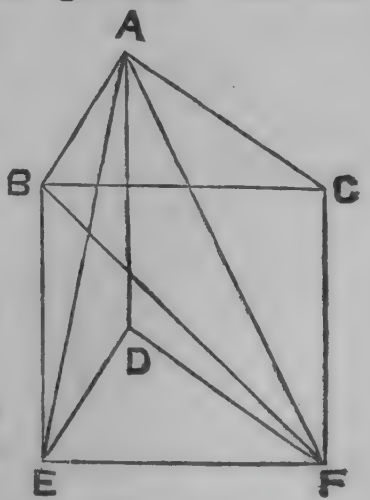


FIG. 133.

$\therefore$ they are equal.

Also ABEF = ABCF $\because$ they have the same altitude, *viz.*, the $\perp^r$ from A on the plane BCEF, and they have equal bases, *viz.*, $\triangle$BEF and $\triangle$BCF each being half the $\square^m$ BCEF.

$\therefore$ the volume of a triangular pyramid $= \frac{1}{3}$ of a right prism of the same base and altitude

$= \frac{1}{3}$ base $\times$ height.

If the base is a ploygon it may be divided into a number of triangles and the same formula derived for that pyramid.

Thus **the volume of any pyramid $= \frac{1}{3}$ the base $\times$ the altitude.**

Another proof. Take the centre of a cuboid and join it to each of its corners. Then since the cuboid has six faces it will be divided into 6 equal pyramids (by symmetry).

$\therefore$ volume of each pyramid $= \frac{1}{6}$ of the cuboid

$= \frac{1}{6}a^3$ where a is a side of the cube

$= \frac{1}{3} \cdot \frac{1}{2}a \cdot a^2 = \frac{1}{3}$ height of the pyramid $\times$ area of the base.

The square pyramid shown in fig. (134) may be regarded as made up of a very large number of square slices such as *abcd*. Imagine that each square is slipped along [as explained in § 110] and then a slanting pyramid of the same volume is got.

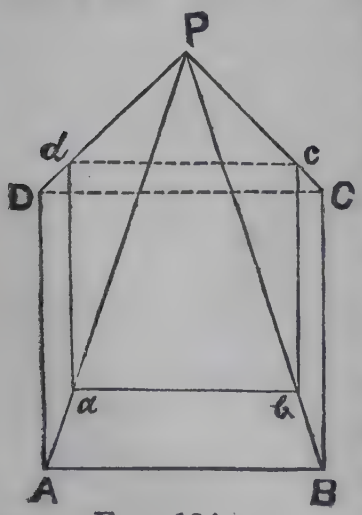


FIG. 134.

In this case also the volume $= \frac{1}{3}$ of the area of the base $\times$ altitude. Reasoning in this way it may be shown that the volume of any square pyramid $= \frac{1}{3}$ (area of base) $\times$ (altitude.

Now to find the volume of a pyramid standing on any polygon, we imagine the polygon to be made up of a number of very small squares and the pyramid to be made up of square pyramids standing on these small squares; thus the volume of a pyramid on any polygon = the sum of the volumes of these square pyramids = $\frac{1}{3}$ × the altitude × the sum of the areas of the squares = $\frac{1}{3}$ the base of the polygon × the altitude.

Experimental proof. Take a small hollow prism, insert a pyramid having the same base and height. Fill up the prism with water, then remove the pyramid and pour water to fill the prism, it will be found that the additional water required is half the original quantity of water; hence the volume of the pyramid is one-third that of the prism.

Ex. The base of a pyramid is a square of 3 ft. edge, and the height is 8 ft. Find its volume.

Area of base = $3 \times 3 = 9$ sq. ft.

$\therefore$ Volume = $\frac{1}{3} \times 9 \times 8$ or 24 cub. ft.

Exercise XIV (a).

1. The base of a pyramid is an equilateral triangle, the side being 4 in. and the height is 9·5 in. Find its volume.

2. The base of a pyramid is an isosceles triangle, whose base is 12 ft., and each of the equal sides is 13 ft. and the height of the pyramid is 6 ft. Find its volume.

3. The base of a pyramid is a square of 3·5 ft. side and heigh 10 feet. Find the total area and volume of the pyramid.

4. The base of a pyramid is a regular hexagon, side 3·75 ft. and its height is 9 ft. Find its volume.

5. The base of a pyramid is a pentagon of 2 ft. side. If the volume is 32·24 cub. ft., what is the height of the pyramid?

§ **112. Mensuration of cone.** If a right-angled triangle revolves round one of its sides containing the right-angle as axis until the triangle returns to its original position, the solid generated is a right circular cone. The fixed side is called the axis of the cone. The end of the axis not the right-angle, is called the vertex of the cone, the hypotenuse of the △ in any position is called a slant side of the cone and the circle which is described by the side adjacent to the right-angle is called the base of the cone.

Surface of a cone. If the curved surface of a cone be completely wrapped by a piece of thin paper, as in the case of the cylinder, so that there is no overlapping, and if the paper is unrolled into a plane, it takes the form of a sector, the circumference of the base becomes the arc of the sector and the slant side of the cone its radius. [See Fig. 135].

∴ the c. s. of the cone = the area of the sector.

= $\frac{1}{2}$ radius of the sector × the arc = $\frac{1}{2}$AC × arc AB

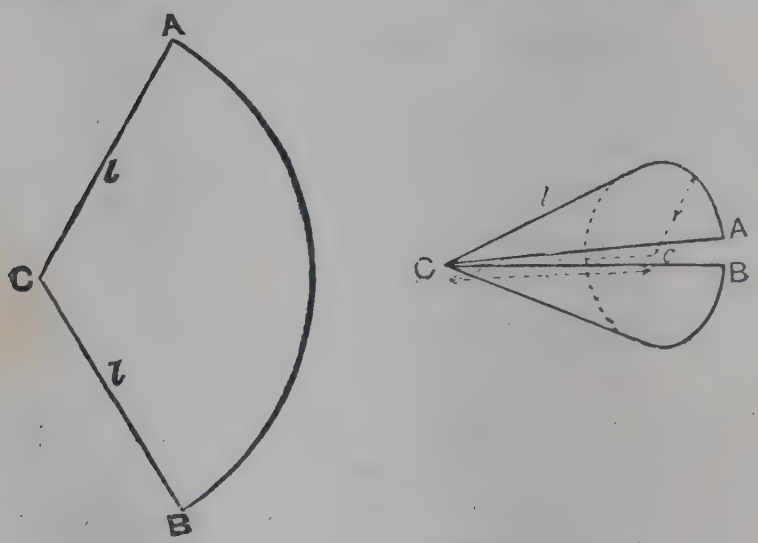


FIG. 135.

$= \frac{1}{2}$ slant side of the cone $\times$ circumference of the base

$= \frac{1}{2}l \times 2\pi r$ (where l is the slant side of the cone and r the radius of the base).

$= \pi rl$ ... ... (1)

$= \pi r . \sqrt{r^2+h^2}$ ... ... (2)

where h is the height of the cone.

[*N.B.*—Instead of wrapping a paper round the cone, a mark at a point A on the circumference of the base may be made and the cone placed on a piece of paper, with the slant side joining the vertex and the marked point A in contact with the paper, the vertex being then kept fixed in a fixed position by one hand, the cone may be rolled until the point A again touches the paper.]

Total surface of the cone $= \pi lr + \pi r^2 = \pi r(l + r)$.

Second proof. The cone may be regarded as the limiting form of a pyramid on a regular polygon, with its vertex directly above the centre of the polygon, when the number of sides of the polygon is indefinitely increased.

The lateral surface of such a pyramid on a regular polygonal base $= \frac{1}{2}$ the perimeter of the polygon $\times$ slant height.

$\therefore$ the curved surface of a cone

$= \frac{1}{2}$ the circumference of the base $\times$ slant height

$= \frac{1}{2} \times 2\pi r \times l = \pi rl.$

Volume of a cone. Volume of a pyramid on a regular polygon, with its vertex directly above the centre of the polygon $= \frac{1}{3} \times$ the height $\times$ area of the base. When the number of sides of the base is indefinitely increased, we get a cone.

$\therefore$ volume of a cone

$= \frac{1}{3} \times$ the height $\times$ area of the circular base

$= \frac{1}{3} \times h \times \pi r^2 = \frac{1}{3}\pi r^2 h$ ... ... (1)

$= \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$ ... ... (2)

where l is the slant height of the cone.

Ex. 1. Find the total surface of a cone, diameter of base 8 ft. and height 3 ft.

Here $r = 4$ ft. and $h = 3$ ft.

$\therefore$ slant height $l = \sqrt{4^2+3^2} = 5$ ft.

$\therefore$ total surface $= \pi r(l + r) = \pi 4(9)$

$= 36 \times 3{\cdot}1416 = 113{\cdot}10$ sq. ft.

Ex. 2. The diameter of the base of a cone is 6 ft., height 8 ft. Find its volume.

Vol. $= \frac{1}{3}\pi r^2 h$ $= \frac{1}{3} \times \frac{22}{7} \times 9 \times 8$

$= \frac{528}{7} = 75{\cdot}43$ cub. ft. nearly.

Ex. 3. Find the lateral surface of a square pyramid side of base 12 ft. and height 8 ft. (the vertex of the pyramid is directly above the centre of the square.)

Here the altitude of a triangular face of the pyramid $= \sqrt{8^2+6^2} = 10$ ft.

$\therefore$ lateral surface = altitude of a face $\times \frac{1}{2}$ the perimeter of the base

$= 10 \times 6 \times 4$ or 240 sq. ft.

Exercise XIV (b).

($\pi = 3{\cdot}1416$.)

1. The diameter of the base of a cone is 3 ft. and the height 6 ft. Find the curved surface of the cone.

2. The slant side of a cone is 140 cm., the base 186 cm. in diameter. Find the total surface.

3. The lateral surface of a cone is 240·3 sq. ft. If the diameter of the base is 162 in., what is the slant height ?

4. The radius of the base of a cone is 4 m., height 50 cm., find its lateral surface.

5. The lateral surface of a cone is 300 sq. ft. If the radius of the base is 6 ft., find (i) the slant height (ii) the vertical height of the cone.

6. A circle of 6 in. radius is divided into five equal sectors; if one of these sectors be folded into the form of a cone, what is the radius of its base and its altitude ?

7. If, from a circle of 5 in. radius, a sector be cut out containing an angle of 192°, what will be the altitude, and the radius of the base of the conical surface formed from it.?

8. Find the total surface of a cone; slant height 8 ft. and radius of base 6 ft.

9. The diameter of the base of a cast iron cone is 8 in. and the height 9 in. What is the volume and weight of the cone? (One cu. ft. of cast iron weighs 7·2 × 1000 oz.).

10. The weight of a cast iron cone is 7·5 ft. If the height of the cone is 5 in., find the diameter of its base.

11. A vessel of concial shape is used to measure petroleum and holds exactly one gallon of it. If the diameter of the base is 12 in., find its height (1 gallon = 277·271 cub. in.)

12. The slant height of a conical tent is 125 in. If the diameter of the base is 160 in., find the volume of air contained within the tent.

13. Find the lateral and total surface of a square pyramid, base 8 ft., height 5 ft.

14. The lateral surface of a square pyramid is 100·6 sq. in. If the side of the base is 6 ft., what is the slant height ?

*§ 113. Frusta of cones and pyramids.

The part of a cone or pyramid intercepted between two planes parallel to its base is called a frustum of a cone or a pyramid as the case may be.

The frustum of a cone may be regarded as generated by the revolution of a trapezium of the form shown in the margin (Fig. 136) about the side which is perpendicular to its parallel sides.

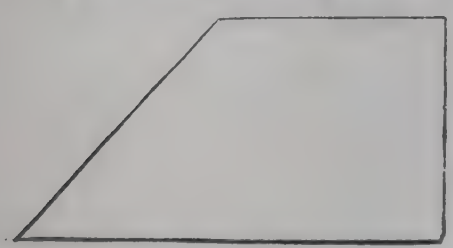


FIG. 136.

If a frustum of a cone be unfolded it will be found that the curved surface is the difference of too similar sectors (Fig. 137).

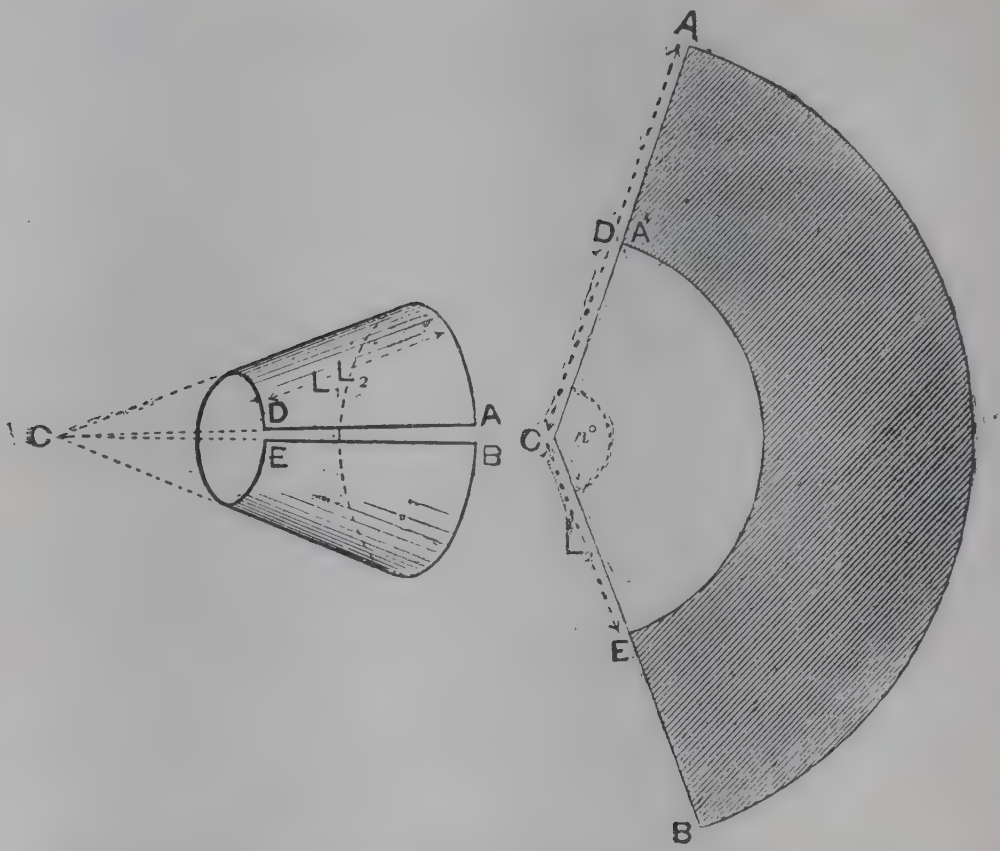


FIG. 137.

If n is the number of degrees in the angle, and r_1 and r_2 are the radii, of the larger and smaller sectors, the area of the curved surface of the frustum

$$= \frac{n}{360} \times \pi r_1^2 - \frac{n}{360} \times \pi r_2^2$$

$$= \frac{\pi n}{360} (r_1^2 - r_2^2) = (r_1 - r_2)(r_1 + r_2) \cdot \frac{n\pi}{360}$$

$$= (r_1 - r_2) \cdot \tfrac{1}{2} \left[\frac{n}{360} 2\pi r_1 + \frac{n}{360} 2\pi r_2 \right]$$

= difference in radii × the mean of the two arcs

= slant height of the frustum × the mean of the circumferences of the ends ... ... (1)

= slant side of the frustum × π × the mean of the diameters of the ends.

$= l \times \pi \cdot \frac{1}{2}(d_1 + d_2)$; where d_1 and d_2 are the diameters of the ends and l is the slant height.

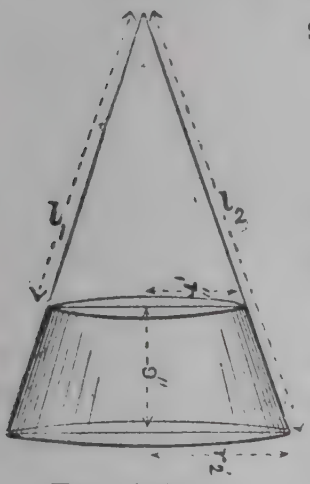


FIG. 138.

2nd proof.

c. s. (curved surface) of the frustum of a cone = difference of the curved surfaces of two cones whose difference is the frustum $= \pi r_1 l_1 - \pi r_2 l_2$ where l_1 and l_2 are the slant heights of the larger and smaller cones, and r_1 and r_2 the radii of the corresponding bases.

By similar $\triangle$s $\frac{r_1}{r_2} = \frac{l_1}{l_2}$. $\therefore\ l_1 = \frac{r_1 l_2}{r_2}$.

$\therefore$ c. s. of the frustum $= \pi r_1^2 \cdot \frac{l_2}{r_2} - \pi r_2 l_2$

$= \pi \frac{l_2}{r_2}(r_1^2 - r_2^2) = \pi \cdot \frac{l_2}{r_2}(r_1 + r_2)(r_1 - r_2)$

$= \pi (r_1 + r_2)\left[\frac{l_2 r_1}{r_2} - l_2\right] = \pi (r_1 + r_2)(l_1 - l_2)$

= mean of the ⊙s × slant height of the frustum.

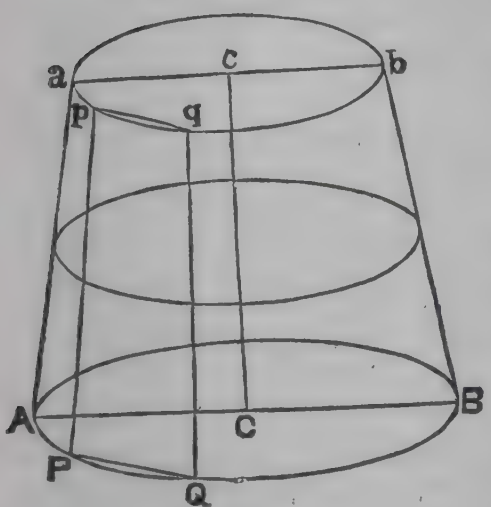


FIG. 139.

3rd proof.

In this figure AB*ba* is a symmetrical trapezium; and the frustum may be regarded as generated by the revolution of CB*bc* about C*c*. Let PQ be a side of a regular polygon supposed inscribed in the circle

generated by the revolution of BC; pq the corresponding side of a similar and similarly placed polygon in the circle generated by bc. PQqp is a symmetrical trapezium.

Its area $= \frac{1}{2}(\text{PQ} + pq) \times \perp^{\text{r}}$ distance between PQ and pq,

and the c. s. of the frustum = the sum of such areas in the limit when the number of sides of the polygon is indefinitely increased.

$\therefore$ ultimately c. s. of the frustum $= Aa \times \frac{1}{2}$ (circumference of the $\odot$ APQB + circumference of the $\odot$ $apqb$) = slant side of the frustum $\times$ mean of the circumferences.

The lateral surface of a frustum of a pyramid on a base in the form of a regular polygon, with its vertex directly above the centre of the polygon, may be found thus.

Each face of the frustum is a trapezium

Its area $= \frac{1}{2}(a + a') \times$ slant height, where a and a' are sides of the two polygonal ends.

$\therefore$ the total lateral surface $= \frac{1}{2}n(a + a') \times$ slant height

= slant height $\times \frac{1}{2}(p + p')$ where p and p' are the perimeters of the polygons

= slant height $\times$ mean of the perimeters.

Now if the number of sides of the regular polygonal ends of the frustum be indefinitely increased, the frustum becomes that of a cone and $\therefore$ the lateral surface of a conical frustum is slant side $\times$ mean of the circumferences of the ends.

*§ **114. Volumes of frusta.** Let the height of the frustum be h and the radii of its ends be R and r. If the cone of which it is a frustum be completed,

and if x be the height of the cone added on in completion of the cone, we have by similar $\triangle$s

$$\frac{x}{r} = \frac{x+h}{R}$$

$$xR = xr + rh$$

$$x(R - r) = rh.$$

$$\therefore \quad x = \frac{rh}{R-r} \quad \ldots \quad \ldots \quad \ldots \ (a)$$

$$\text{and } x + h = \frac{Rx}{r}$$

$$= \frac{R}{r} \times \frac{rh}{R-r}$$

$$= \frac{Rh}{R-r} \quad \ldots \quad \ldots \quad \ldots \ (b)$$

Volume of the frustum of a cone

= difference of the volumes of the whole cone and the conical part which is introduced in completing the cone

$$= \tfrac{1}{3}\pi R^2 . (x + h) - \tfrac{1}{3}\pi r^2 . x$$

$$= \tfrac{1}{3}\pi R^2 . \frac{Rh}{R-r} - \tfrac{1}{3}\pi . r^2 . \frac{rh}{R-r} \text{ from } (b) \text{ and } (a)$$

$$= \tfrac{1}{3} . \frac{\pi h}{R-r} \{ R^3 - r^3 \}$$

$$= \tfrac{1}{3}\pi h . \{ R^2 + Rr + r^2 \} \quad \ldots \quad \ldots \ (1)$$

$$= \frac{h}{3} \{ \pi R^2 + \pi Rr + \pi r^2 \}$$

$$= \frac{h}{3} \{ \pi R^2 + \sqrt{\pi R^2 . \pi r^2} + \pi r^2 \}$$

$$= \frac{h}{3} \{ S + \sqrt{Ss} + s \} . \quad \ldots \quad \ldots \quad (2)$$

where S and s are the areas of the ends of the frustum.

Similarly if S and s represent the areas of the ends of a

frustum of *any* pyramid and h its height, its volume may be shown to be $\frac{h}{3}(S + \sqrt{Ss} + s)$

If a_1, a_2 be the corresponding sides of the polygonal ends of the frustum we have $\frac{a_1}{a_2} = \frac{x}{x+h}$ where $x + h$ is the height of the whole pyramid of which this is a frustum.

$\therefore \quad x(a_2 - a_1) = a_1 h.$

$\therefore \quad x = \frac{a_1 h}{a_2 - a_1}$ and $x + h = \frac{a_2 h}{a_2 - a_1}$ as before.

Volume of the frustum $= \frac{1}{3}S \cdot (x + h) - \frac{1}{3}s\,(x)$

$= \frac{1}{3}S \cdot \frac{a_2 h}{a_2 - a_1} - \frac{1}{3}s \cdot \frac{a_1 h}{a_2 - a_1}.$

Also $\frac{S}{s} = \frac{a_2^2}{a_1^2}$ (areas of similar figures are as the squares of corresponding sides).

$\therefore \quad V = \frac{1}{3} \cdot s \frac{a_2^2}{a_1^2} \cdot \frac{a_2 h}{(a_2 - a_1)} - \frac{1}{3}s \frac{a_1 h}{a_2 - a_1}$

$= \frac{1}{3} \frac{s \cdot a_1^3 h}{a_1^2 (a_2 - a_1)} - \frac{1}{3} \frac{s a_1 h}{(a_2 - a_1)}$

$= \frac{1}{3} \frac{s \cdot h}{a_2 - a_1} \left\{ \frac{a_2^3 - a_1^3}{a_1^2} \right\}$

$= \frac{1}{3} \frac{s \cdot h(a_2^2 + a_1 a_2 + a_1^2)}{a_1^2}$

$= \frac{h}{3} \left[\frac{s a_2^2 + s a_1 a_2 + s a_1^2}{a_1^2} \right]$

$= \frac{h}{3} \left[S + s \frac{a_2}{a_1} + s \right] = \frac{h}{3} \left[S + s \cdot \sqrt{\frac{S}{s}} + s \right]$

$= \frac{h}{3} \left[S + \sqrt{Ss} + s \right].$

Ex. (1). The edges of the parallel ends of the frustum of a square pyramid are 30 in. and 18 in. respectively, the height of the frustum is 10 in. Find the volume of the frustum.

$$\text{Volume} = \frac{h}{3}(S + \sqrt{Ss} + s)$$

$$= \frac{10}{3}(900 + \sqrt{30^2 \cdot 18^2} + 324)$$

$$= \frac{10}{3}(900 + 540 + 324)$$

$$= \frac{10}{3} \times 1764 = 5880 \text{ sq. in.}$$

Ex. (2). The diameters of the ends of a conical frustum are 20 in. and 10 in. and the distance between them is 14 in. Find the curved surface of the frustum and also its volume.

c. s. = Slant side $\times \pi \times$ mean of the diameters.

(Slant side)$^2 = 14^2 + 5^2 = 196 + 25 = 221$ sq. in.

(Slant side) $= \sqrt{221} = 14{\cdot}87$ in.

$$\therefore \text{ c. s.} = 14{\cdot}87 \times 3{\cdot}142 \times \frac{20 + 10}{2}$$

$$= 14{\cdot}87 \times 3{\cdot}142 \times 15 \text{ sq. in.}$$

$$= 700{\cdot}82 \text{ sq. in.}$$

$$\text{Volume} = \pi\frac{h}{3}(R^2 + Rr + r^2)$$

$$= \frac{3{\cdot}142}{3} \times 14\,(20^2 + 20 \times 10 + 10^2)$$

$$= \frac{3{\cdot}142}{3} \times 14 \times 700$$

$$= \frac{314{\cdot}2 \times 98}{3} = 10264 \text{ cub. in.}$$

Exercise XIV (c).

1. A metal plate basin is in the form of a frustum of a cone, the diameters of the upper and lower ends are 14 in. and 10 in. respectively, the slant edge is 4 in. How many square inches of the metal plate does it contain? Find the volume of the frustum.

2. Find the curved surface and volume of each of the following conical frusta :—

	Diameter of the upper face.	Diameter of the lower face.	Height.
(*a*)	2 cm.	16 cm.	5 cm.
(*b*)	12·3 in.	18·9 in.	5·4 in.
(*c*)	75 in.	95 in.	24 in.

3. Find the volume of the frusta of the following pyramids :—

(*a*) The upper and lower faces are squares of sides 4 ft. and 6 ft. respectively, and the height is 8 ft.

(*b*) The upper and lower faces are equilateral $\triangle$s of sides 5·3 in. and 3·8 in. respectively, and the height is 10 in.

(*c*) The upper and lower faces are regular hexagons whose sides are 10 in. and 8 inches respectively and the height is 10·5 inches.

4. The ends of the frustum of a pyramid are squares of 40 and 60 in. If it is 10 inches thick, find its slant surface. [The line joining the centres of the squares is perpendicular to the ends.]

5. The ends of a frustum of a pyramid are rectangles ; the base measuring 10 by 8 in. and the top 4 by $2\frac{1}{2}$ in. If the thickness is 4 inches, find the volume.

6. The ends of the frustum of a pyramid are triangles the sides of base measuring 26 in., 24 in., 10 in., and the sides of the top $3\frac{1}{4}$ in., 3 in., $1\frac{1}{4}$ in. If the thickness of the frustum is 7 in., find the volume.

§ **115. Sphere.** If a semi-circle revolve about its diameter as axis until it returns to its former position, the solid figure generated is called a sphere; the centre and the radius of the semi-circle are called the centre and radius of the sphere.

A frustum or zone of a sphere is the solid intercepted between two parallel planse.

Surface of a sphere. Let O be the centre of a semi-circle and AOB a diameter about which it rotates to form the sphere. (Fig. 140). Let PQ be a side of a regular polygon inscribed in the circle and PpqQ be a small trapezium which generates the frustum of a cone as the circle rotates about AB. Draw ON ⊥ to PQ. Then PQ is bisected at N. From N draw Nn ⊥ to AB.

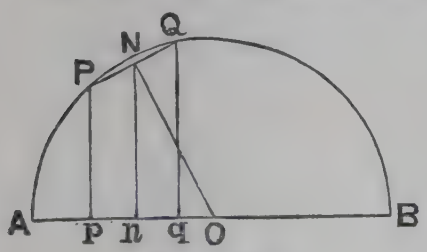


FIG. 140.

The curved surface of the frustum = PQ × 2πNn (2Mn being the mean of the diameters of the ends).

By similar triangles $\frac{PQ}{pq} = \frac{ON}{Nn}$

$\therefore$ PQ. Nn = pq. ON

= R. pq.

(where R is the radius of the circle, therefore also of the sphere)

$\therefore$ c. s. of the frustum = 2π. R. pq.

The surface of the sphere is the limit of the sum of the curved surfaces of the frusta corresponding to the sides of the polygon when the number of sides is indefinitely increased.

$\therefore$ surface of the sphere

= 2πR × the sum of the lengths pq

$= 2\pi R \times$ the diameter of the sphere

$= 4\pi R^2$.

The proof is applicable to the surface of a zone and hence.

$\therefore$ the surface of a zone $= 2\pi Rd$.

where d is thickness of the zone.

The curved surface of the circumscribing cylinder (Fig. 141) of the sphere $= 2\pi R \cdot 2R = 4\pi R^2$.

$\therefore$ the surface of a sphere = the c. s. of the circumscribing cylinder.

Experimental proof—Strip the cover from a tennis ball, place the pieces on squared paper and trace the outline. The area may be found by counting the squares or by applying Simpson's rule. Or the outline of the cover traced on cardboard may be cut out and weighed and thus the area ascertained.

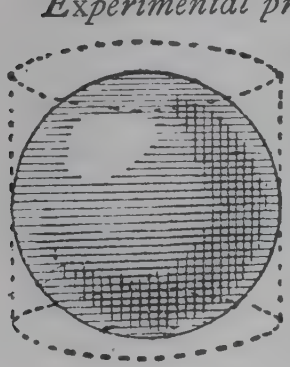


FIG. 141.

The area will then found to be $4\pi R^2$.

Volume of a sphere—An element square may be drawn on the surface of a sphere and its angular points joined to O. (Fig 142). A small pyramid will thus be formed and its volume $= \frac{1}{3}$ (area of the element) $\times$ radius of the sphere. The whole surface of the sphere may be supposed to be divided into a very large number of

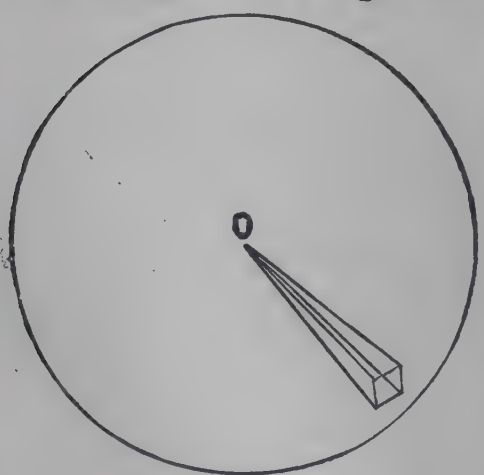


FIG. 142.

such element areas. Then the sphere will consist of a number of pyramids. ∴ volume of a sphere = $\frac{1}{3}$ surface of the sphere × radius of the sphere = $\frac{1}{3} \times 4\pi r^2 \times r = \frac{4}{3}\pi r^3$.

Ex. 1. What is the volume of a spherical shell external diameter 10 in., internal diameter 6 in.?

The volume = difference between the volumes of spheres of radii 5 in. and 3 in. respectively

$= \frac{4}{3}\pi . 5^3 - \frac{4}{3}\pi . 3^3$

$= \frac{4}{3}\pi \times 98$

$= \frac{4}{3} \times 3{\cdot}1416 \times 98$

$= 4{\cdot}1888 \times 98 = 410{\cdot}5$ cub. in.

4·1888
98
376·99
33·50
410·49

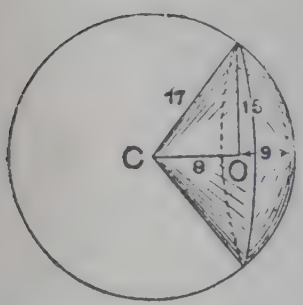


FIG. 143.

Ex. 2. What is the total surface of a spherical sector obtained by joining the centre of a sphere 17 in. in radius to the points on the circumference of a small circle of 15 cm. radius drawn on its surface? Find also its volume.

The distance of the small circle from the centre $= \sqrt{17^2 - 15^2} = 8$ in.

∴ the thickness of the corresponding segment (spherical cap) = 17 − 8 = 9 in.

∴ c. s. of the segment $= 2\pi rd$

$= 2 . \pi . 17 . 9 = 306\pi$.

c. s. of the cone resting on the small circle $= \pi rl = \pi . 15 . 17 = 255\pi$

∴ the total surface of the sector

$= (306 + 255) = 561\pi = 1762$ sq. in.

3·1416
561
15708
1885
31
1762·4

Volume of the sector $= \frac{1}{3} \times$ radius $\times$ (area of c. s. of the spherical cap)

$= \frac{1}{3} \times 306\ \pi \times 17$

$= 102\ \pi \times 17 = 1734\ \pi$

$= 5448$ cub. inches.

3·1416
1734
3141·6
2199·1
942
126
5447·5

Ex. 3. In the previous example, find the volumes of the two portions (called segments) into which the sphere is divided by the small circle.

The volume of the smaller segment.

= the volume of the sector—the volume of the cone.

$= 1734\pi - \frac{1}{3} \times \pi r^2 \times h$

$= 1734\ \pi - \frac{1}{3} \times \pi\ (15)^2 \times 8$

$= 1734\ \pi - 75 \times 8\ \pi$

$= 1734\ \pi - 600\ \pi$

$= 1134\ \pi$

$= 3563$ cub. inch.

3·1416
1134
31416
3142
942
125
3562·5

N.B.—If there are two small circles, the volume of the intercepted portion AA′BB′ can be found by taking the difference of the volumes of the spherical segments A′CB′ and ACB.

Exercise XIV (d).

($\pi = 3{\cdot}1412$.)

1. Find the volume and weight of a cannon ball 9 in. in diameter [s. g. of metal = 7·2]

2. The diameter of a gold sphere is 2·5 inches. Find its weight, if the specific gravity of gold is 19.

3. The external diameter of a cocoanut shell is 4 inches and the internal diameter 3 in. Find the quantity of cocoanut in the shell and the water in the hollow portion supposing that the hollow portion is only $\frac{3}{4}$ full of water.

4. Find the internal diameter of a hemispherical bowl holding a quart of water.

5. If 28 cub. inches of powder weigh 1lb., find the weight of powder required to fill a spherical shell whose internal diameter is 6 inches.

6. If the earth is regarded as a sphere having a radius of 4000 miles in length, what is its surface? Find its volume in the form $K \times 10$ cubic miles

***7.** The surface of the earth is divided into 360 equal parts by 360 meridians drawn from the north pole to the south pole. Find in square miles the surface of the earth between longitudes.

1. 45° E & 36° W.
2. 125° 10′ E & 175° 10′ E.

***8.** Calculate in square miles, to 4 significant figures, the surface of—

1. Each of the Frigid zones.
2. Each of the Temperate zones.
3. The Torrid zones

given that the radius of the earth = 4000 miles;
Sin $23\frac{1}{2}°$ = ·3987; Sin $66\frac{1}{2}°$ = ·9171.

9. Thirty small shot are dropped into a burette containing water and there is a rise in the surface of water showing an increase of volume of 425 c.c; find the average diameter of the shot.

10. A sphere of 5 in. radius is cut into two parts by means of a plane distant 3 inches from the centre. Find the volume of the smaller of the two portions.

11. A spherical sector is obtained from a sphere of $7\frac{1}{2}$ in. radius; and the spherical cap bounding it has a height $4\frac{1}{2}$ in. What is the volume of the sector?

12. The interior of an earthen jar has the form of a portion of sphere; the upper face of the jar is 7 inches—the base 5·6 in. in diameter; what volume of water could the vessel hold, its height being 2·1 inches?

13. The inside of an earthen jar is in the form of a portion of a hemisphere of 10 in. radius, with the radius of the base 6 in. What is its total inside curved surface?

14. A double convex lens has two spherical surfaces both of 40 cm. radius and its central thickness is 1·5 cm. What is its weight the density being 2·5.

15. A iron dumb-bell consists of two spheres each 4 in. in diameter, centres being 16 in. apart connected by a bar cylindrical in form $\frac{3}{4}$ in. in diameter. Find its volume and weight.

16. Hydrogen is collected in an inverted flask the lower portion

FIG. 144.

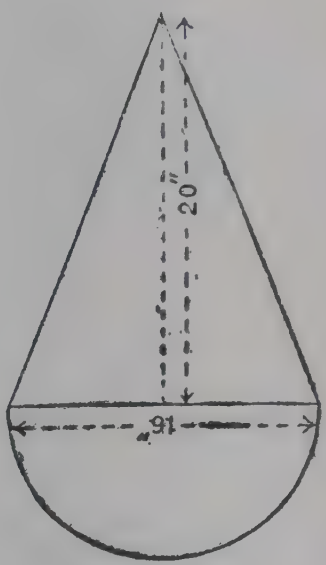


FIG. 145.

of which (Fig. 144) is in the form of a sphere of diameter 13 cm. and the upper portion in the form of a cylinder (diameter 5 cm.). The total height of the flask is 18 cm.; supposing the flask is full of the gas, find the quantity of gas collected in it.

17. A kite consists of a semi-circle and a triangle. The total length of the kite = 30 inches: the greatest breadth = 18 in. Find the area of the kite. (Fig. 145.)

18. The silk covering of an umbrella forms the surface of a spherical segment of 4 ft. radius, and the height of the spherical

segment is 1½ ft. Find the quantity of silk required for the umbrella.

19. A garden tent is in shape a cube surmounted by a pyramid. If the edge of the cube is 8 ft. and the height of the centre pole 12 ft., find (1) the amount of canvas used in making the tent and (2) the amount of air space included in it.

20. A candle-stand chimney is made of mica and is in the form of two conical frusta. The upper one has its upper rim 3 in. in diameter and lower rim 4·8 in; the lower frustum has its upper rim 4·8 in. and its lower 2 in. in diameter. The total axial distance is 5 in. while the heights of the two frusta are in the ratio of 3 : 2. What amount of mica is required? Answer to 1 place of decimals.

21. A candle consists of (1) an upper conical portion ·8 of an inch in height, (2) a lower cylindrical portion. The total length of the candle is 8·2 in. and the greatest width is ·7 inch. Find the volume of the candle.

22. A cane basket used for storing soiled cloths consists of (1) an upper conical frustum; upper rim diameter 12 in., lower rim diameter 3 ft.; (2) a lower cylindrical portion, diameter 3 ft. The total height of the frustum is 4 ft. 4 inches and the slant side of the frustum is 20 inches. Find the cost of making it at 4 as. per sq. foot

23. A wooden cask consists of two equal frusta of a sphere, whose diameter is 3 ft., their broad bases coinciding. The diameters of the ends of each of the frustum are 3 ft. and 1½ ft. Find the height of the cask and the cubical contents of the cask.

24. A municipal lantern is in the form of a frustum of a square pyramid of 1½ ft. in height, a side of the top of which is 20 in. and a side of the bottom 10 in.. Find the lateral surface of the lantern.

25. A brass tumbler is in the form of a frustum, upper rim 8·6 in. diameter, lower rim 3·8 in. diameter, slant side 4 in. Find the quantity of brass required and the cost of coating the inside with lead at 2 annas per square foot.

26. A 56 lb. weight is made in the form of the frustum of a square pyramid, lower edge 5·7 in., upper edge 5·2 in. Find its height, the sp. gr. of the metal is 7·2.

27. A 28 lb. weight is made in the form of the frustum of a cone. If it is of the same height as the weight in the preceding question, find its mean diameter.

28. When a sphere is immersed completely it is found to displace 648 gallons of water, the diameter of the sphere is $3\frac{1}{2}$ in. Find the value of π.

29. A cylindrical boiler has hemispherical ends; its total length is 18 ft. diameter 5 ft. (inside measurement). Find its cubical contents.

30. A stack of coal is in the form of a frustum of a rectangular pyramid, the base is 54 ft. by 24 ft., the top is 36 ft. by 16 ft. and the height is 12 ft. Find the quantity of coal in the stack.

CHAPTER XV.

STOCKS AND SHARES.

§ **116. Stock** is a term applied to the capital of trading companies or the money borrowed by Government to defray the expenses of a nation.

" The promoters of a commercial concern usually originate it publicly by the issue of a prospectus explaining the nature and object of the undertaking, specifying the capital required, the profit likely to be made, and the number of equal shares into which the capital is divided,and inviting the public to become **shareholders.** If sufficient applications be made, the shares are then **allotted** among the applicants, and part, at least, of the amount of each share having been paid, business is begun. Not unfrequently a share is £100, and part of it is paid on application, part on allotment and other parts are called for at such time afterwards as may be thought best, until it is fully **paid up**. Before shares have been fully paid up, it would be troublesome to deal in fractional parts of a share, and consequently only whole numbers of shares are sold ; but afterwards this drawback, in a great part, disappears, it being found almost as convenient that a person should own £325 or £456 of the capital of a company, as that he should own £300, £400 or £500. It is consequently common to have arrangements made for this, and then **shares** are no longer spoken of, but **stock**, and any portion of the capital which is not less than £1 can be transferred from one person to another. Thus 7 hundred-pound shares will be represented by £700 of the stock or

capital, or shortly by £700 stock ; 12 shares, if ten-pound shares, by £120, and so on."

A **scrip** is a preliminary certificate of a subscription to the capital of a bank, railway or other company, and contains the amount of the subscription and the dates of the payments of the instalments. When all the instalments are paid, the scrip is exchanged for a share certificate or **bond.**

A stock or share certificate is a document given by the Company to the owner of stock or shares stating the amount of stock or shares owned.

The.....Company, Limited.

This is to certify *that*.....
is a registered holder of.....*shares*
of £.....*each, numbered*.....
inclusive, in the abovenamed Company
and that the sum of £.....*has been*
paid up on each of the said shares.

Given under the seal of the said Company this 1st day of July 1909.

.....
.....
} *Directors.*

.....*Secretary.*

Seal of Company.

The net annual profits of a public company are divided among the shareholders, and the proportion of profits which the shareholders receive according to the shares they possess in the capital is called the **dividend.**

Shares in the capital of a company may be bought or sold like any commodity. Thus a shareholder who subscribed £100 for each share of a company may, if he deems it necessary, sell his share to any one who is desirous of buying it. The price fetched by the share may be £100, or a higher or a lower sum. If it be sold for £100 (the money originally paid for it), then the value is said to be **par.**

Should the prospects of a company be good and the undertaking yield a percentage larger than can be obtained elsewhere, outsiders would like to become shareholders, and the shares would consequently rise in value. Thus, if a share of £100 brings an annual income of 8 % and if the rate of interest in the market be only 5 %, the value of the share would naturally rise nearly in the ratio of 8 : 5, and would become equal to, say, £100 × $\frac{8}{5}$ or £160. In this case the £100 share is said to be at **160*** or simply at **60 premium** or **above par.**

Should money, on the other hand, be dear and the rate of interest in the market be something like 10 % the share would fall in value, and a £100 share would be worth only, say, £100 × $\frac{8}{10}$ or £80. In this case the £100 share is said to be **at 80** or **at 20 discount** or **below par.** This explains *how shares come to be sold at prices differing from the original value.*

* In the phraseology relating to this subject, **a number alone** is used in stating the price, pounds or rupees being the denomination which is understood. When the prices are published in the newspapers, etc., they are generally said to be **quoted.**

The most important point to be remembered in connection with stocks is that *stock is not cash* (£100 stock does not mean £100 cash) and that *stocks can be bought and sold just like any other commodity the price varying day by day and even hour by hour.*

But the student should note that *there can be no definite ratio whatever between stock and cash*, as the prices of stocks are fluctuating owing to an almost infinite variety of causes, such as the abundance or scarcity of money and the opportunities of employing it to advantage in mercantile speculations, to the rumours of a new loan or of the imposition of a fresh tax, or even the repeal of a tax; to rumours of war and to innumerable other causes relative to the trade, finance and other domestic affairs of the country. Therefore, if a £100 share yield an annual income of £3, *it does not necessarily follow* that a £50 share in another concern brings an income of £$1\frac{1}{2}$.

Sometimes a portion of the capital required is raised on different conditions from the rest, *viz.*, by an agreement to pay a fixed rate of dividend, even if there be nothing left for the ordinary subscribers, or by getting another company or the Government to guarantee the payment of the fixed rate of dividend. Such shares are called *preference* or *guaranteed* shares.

Guaranteed shares are shares the dividend on which must be paid by the company which guarantees the payment, if it cannot be paid by the company that owns the shares.

Preference shares are shares on which the dividend is paid as long as the net annual income of the concern is sufficient, even if there is nothing left to pay anything to the ordinary shareholders. They have *prefer-*

ence over ordinary stock not only with regard to dividend but also on liquidation.

§ **117.** The sum of money borrowed by the Government of a country is called the **national debt**. When money far beyond what the ordinary revenues of the country can provide is required by the Government and when it is considered imprudent to impose a fresh tax, the Government is driven to borrow money upon interest quite like a private individual. The Government is then said to issue a loan. Even last July (1913) the Government of India raised a loan of 3 crores of rupees by issuing the following Notification :—

"Promissory notes will be issued for the said amount in the form * * of the notes of the *three-and-a-half per cent.* loan of 1900-01, of which loan the notes to be now issued will form a part * * . The interest on the notes of the loan is payable half-yearly on the 30th day of June and the 31st day of December. Tenders for the whole or any part of the said amount of 3 crores will be received by the Comptroller-General, Calcutta * * . Tenders must be for sums of 100 Rupees or multiples of that sum * * . The rate at which each tender is made must be specified in rupees or rupees and annas * * . The minimum rate at which tenders will be accepted will be recorded under the signature of the Comptroller-General, and, before the tenders are opened, placed upon the table in a sealed envelope, but will not be declared unless some tender is rejected only because it is below the recorded minimum * * . Tenders at rates not below the minimum rate will be accepted in the order of the rates tendered, beginning with the highest rate * * ."

Often it happens that the Government can never hope to repay the amount borrowed, and what is usually done in such a case is to **fund** the debt. The operation of *funding* consists in converting the debt into a permanent loan, the interest on which is to be paid out of the funds or the permanent revenues of the country. This term

(funds*), which *originally* signified the taxes or funds set apart as a security for the repayment of the debt and the interest thereon, is generally applied now to the national debt itself. Any person who owns a portion of the funded debt of a country is called a **fund-holder**, and he is furnished by the Government with an acknowledgment of the debt which entitles him to the interest when due. The money advanced to the Government, is not to be paid on demand. Therefore the fund-holder who requires ready cash cannot not claim it from the Government, but must find a purchaser of the paper in the stock market. What he offers for sale may be considered to be nothing more than a right to a fixed annual income † the payment of which, till the repayment of the principal, is guaranteed on the security of the Government. The price he may get in the market will depend upon the abundance or scarcity of money in circulation and various other causes.

Printed certificates acknowledging the indebtedness of the Government to the owners of the certificates are known as **bonds** or **debentures.** Debenture holders receive a fixed rate of interest.

If a bond of a company, &c., contains a promise to pay the principal of the bond, it is called a **terminable bond;** if otherwise, it is known as a **perpetual bond.**

---

* In 1751, the Government of England united the various loans into one fund called the **consolidated fund** at the uniform interest of 3 per cent. And since *Government stocks are named from the fixed percentage attached to them*, the sums due in this consolidated fund were termed '**the 3 per cent. consols.**' The rate of interest has now been reduced to 2½ per cent.

† This invariable income is called a *dividend*, after the language of the joint-stock companies, though there is actually no division of profits.

The **nominal value** of a bond is the principal or the amount printed on the bond. The amount for which it is bought or sold is its cash price or value.

NOTE 1.—A scrip or bond for £100 remains of the *nominal* value of £100 in spite of any alteration in the market value of the bond. Its *real* value is the price which it will obtain, if put up for sale in the public market.

NOTE 2.—£100 stock is sometimes known as one *cent.* of that stock. It is adopted as the most convenient unit in stock transactions.

NOTE 3.—The difference between stock and shares must be clearly borne in mind. While *any* portion of the stock of a Railway Company or of the Funds may be bought, sold or held, Banks and Trading Companies permit only the holding, buying and selling of *complete* shares. Thus it is possible for one to hold £347 of the stock of a Railway Company. But he may hold only Rs. 500 (or integral multiples of Rs. 500) of the stock of the Bank of Madras, the nominal value of each of whose shares is Rs. 500.

§ **118.** No transaction in the stock market can be made except through the medium of a broker. A **broker** is an agent who sells and buys stock for the public at a fixed commission, and the percentage paid to him is called **brokerage.** The broker acting for his client deals with the class of people known as **stock-jobbers or stock-dealers.** These latter are speculators who deal in the purchase and sale of stocks, buying when the price is low to sell when it rises. When the price of any stock, say the 3 per cent. consols, is quoted as $85\frac{1}{2}$—$86\frac{1}{2}$, it is to be understood that jobbers are prepared to buy at the lower price and sell at the higher. An extract of the quotation as it generally appears in the newspapers is given overleaf.

MONEY MARKET.

MADRAS, August 15th, 1912.

3½ p. c. Government Paper	... Rs. 96 to Rs. 96/12
3 p. c. Government Paper	... Rs. 81/8 to 82/8
Bank of Madras Shares (Rs. 500)	... Rs. 1,480 to 1,500.

EXCHANGE.

Bank Telegraphic Transfer	... ¼
Bank Demand	... ¼ 1/32
3 Months' sight credits on London	... ¼ 1/4
6 Months' sight credits on London	... ¼ 7/16
3 Months' sight D/P on Marseilles Fics	... Fics 171 1/2
60 D/S D/P Japan	... Rs. 150
Tone of Exchange :—Quiet	
Bank Rate	... 4 %

BOMBAY, August 15th, 1912.

Bank Telegraphic Transfer	... ¼
Bank Demand	... ¼ 1/32
3 Months' sight credits on London	... ¼ 1/4
Bank Rate	... 3 %

TRANSACTIONS.

Nil.

SCOTT AND TOD.

NOTE 1.—The commission paid to a stock-broker is *always* calculated on the *amount of stock* that he buys or sells, and *not on the price* at which it may be bought or sold.

NOTE 2.—In the sale and purchase of Government stock, brokers charge commission at $\frac{1}{8}$ per cent. in England and $\frac{1}{16}$ per cent. in Madras.

NOTE 3.—The seller of stock will receive the sale money—broker age, while the purchaser has to pay purchase-money + brokerage.

Thus, if the price of £100 stock is £84, and brokerage $\frac{1}{8}$ per cent., a person buying stock has to pay £$84\frac{1}{8}$ for every £100 stock; and a person selling stock has to receive £$83\frac{7}{8}$ for every £100 stock.

NOTE 4.—'The 3 per cents. at $85\frac{3}{8}$, $85\frac{5}{8}$, $85\frac{1}{2}$ means that the price of the stock began at $85\frac{3}{8}$, rose to $85\frac{5}{8}$ and sank when the market closed to $85\frac{1}{2}$.'

NOTE 5.—In the exercises relating to this part of the subject, brokerage need not be allowed unless it is specially mentioned. It is ordinarily included in the price of stocks.

[NOTE 6.—There are separate classes of brokers for several kinds of business to be transacted.

Bill-brokers or Exchange-brokers are persons who engage, either on their own account or that of their employers, in the purchase and sale of inland and foreign bills of exchange (see Art. 113) and promissory notes. These are to be distinguished from:

Discount-brokers who are persons that employ the spare money of capitalists in discounting bills of exchange which have some time to run before they become due.

Stock-brokers (or *share-brokers*), buy and sell stocks in the public funds (and shares in the capitals of joint-stock companies)].

The annual dividend due on Government stock is paid half-yearly and the value of the stock will of course be somewhat greater immediately before, than immediately after, a payment of the dividends.

§ 119. Use of Graphs in Stocks. It is a good exercise in stocks to represent by a graph the fluctuations in the market value of a stock; as an example, the fluctuations (given in the following table) in the market value of $2\frac{1}{2}$ % consols are represented by a graph in (Fig. 146).

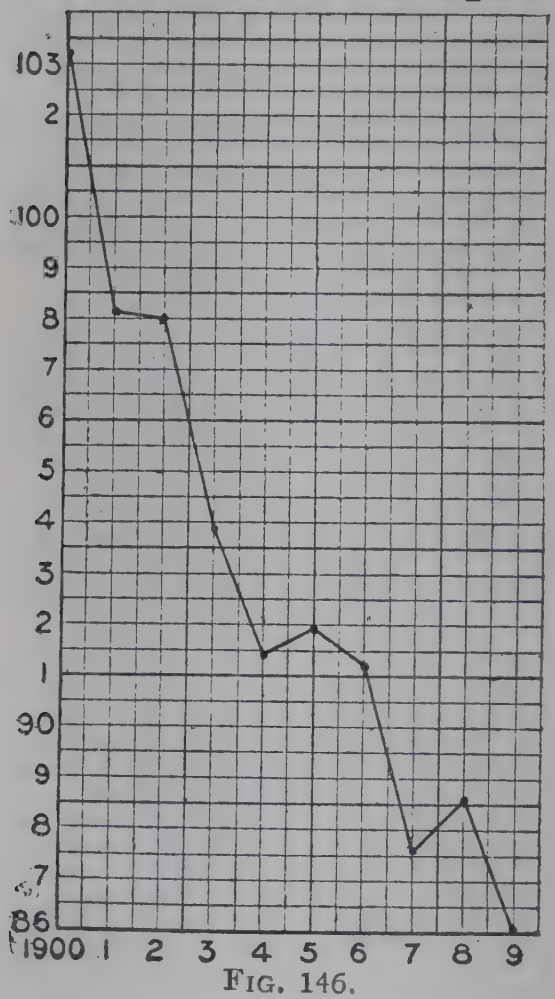


FIG. 146.

CONSOLS $2\frac{1}{2}$ %.

Year.		Highest price.
1900	...	$103\frac{1}{4}$
1901	...	$98\frac{1}{8}$
1902	...	98
1903	...	$93\frac{7}{8}$
1904	...	$91\frac{7}{16}$
1905	...	$91\frac{15}{16}$
1906	...	$91\frac{1}{8}$
1907	...	$87\frac{9}{16}$
1908	...	$88\frac{9}{16}$
1909	...	83

The following is another example where graphs may be profitably employed.

Find by drawing a graph which of the following will form the best investment. 3 % at 80 or 4 % at 90 or 5 % at 120.

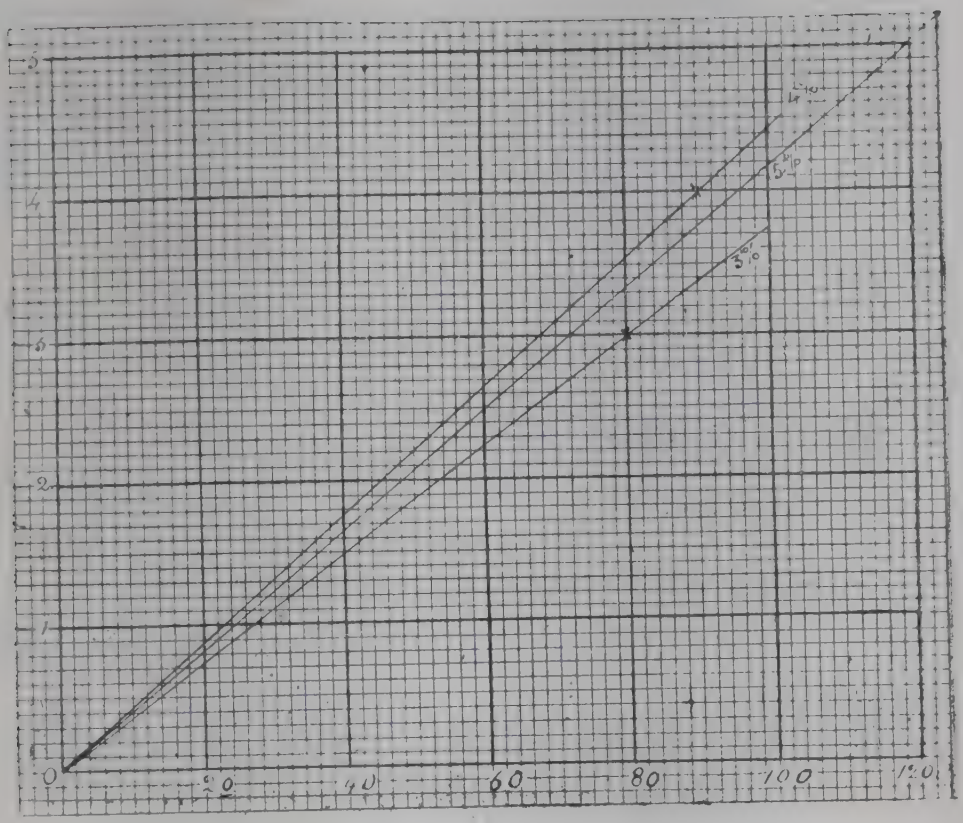


FIG. 147.

In Fig. (147) each vertical big division denotes 1 %
and each horizontal big division denotes 20.

Thus the least inclined of the sloping lines in the figure is the graph of "3 % at 80" &c.

3 % at 80 yields from figure $3\frac{3}{4}$ % interest.

4 % at 90 ... ... $4\frac{1}{2}$ % nearly.

5 % at 120 ... ... $4\frac{1}{6}$ %.

Thus the second is the best investment.

Note. In going through the working of the following questions, the student should carefully distinguish between **cash** and **stock**. *Cash* is money invested, while *stock* is the property obtained by the investment of *cash*.

Ex. 1. What sum of money will purchase £7,000 stock in the 3 per cents. at 98?

Sol.—The money needed to purchase £100 *stock* is £ 98 *cash*.

∴ the money needed to purchase £7,000 stock is 70 × £98 or £6,860 cash.

Ex. 2. Find the stock bought for £700 in the 5 per cents. at 84.

Sol.—With £84 *cash* we can purchase £100 *stock*.

∴ with £700 *cash* we can buy (£700 × $\frac{100}{84}$ or) £833$\frac{1}{3}$ stock.

Ex. 3. Find the yearly income arising from the investment of £720 in the 4 per cents. at 95$\frac{7}{8}$, brokerage being $\frac{1}{8}$ per cent.

Sol.—Now, £100 stock costs £95$\frac{7}{8}$ + the brokerage, that is costs £96 and brings an income of £4.

Thus, an investment of £96 *cash* yields an annual income of £4.

Hence the investment of £720 *cash* brings an annual income of £4 × $\frac{720}{96}$ or £30.

Ex. 4. What sum must be invested in the 5 per cents. at 20 discount to produce £80 a year?

Sol.—To produce £5 annual income, £100 − £20 or £80 cash must be invested.

Hence to produce £80 annual income, £80 × $\frac{80}{5}$ or £1,280 must be invested.

Ex. 5. A person invested £475 in the 4 per cents. at 94$\frac{7}{8}$ and sold out his stock when they had risen at 96$\frac{5}{8}$. Find his gain, brokerage being $\frac{1}{8}$ per cent.

Sol.—£100 stock costs £$94\frac{7}{8}$ + £$\frac{1}{8}$ (brokerage) or £95.

Again by selling £100 stock, he realises £$96\frac{5}{8}$ − £$\frac{1}{8}$ (brokerage) or £$96\frac{1}{2}$.

Now, £95 is the amount required to purchase £100 stock,

$\therefore$ £476 will purchase £$\frac{100 \times 475}{95}$ or £500 stock.

Money realised by selling of £500 stock at $96\frac{1}{2}$ for every £100 $= 96\frac{1}{2} \times 5 =$ £$482\frac{1}{2}$.

$\therefore$ his gain = £$482\frac{1}{2}$ − £475 = £7 10*s*.

Ex. 6. How much more stock would a person who invests £918 in the $3\frac{1}{2}$ per cents. at 90, have obtained, if he had waited till they had fallen to 85?

Sol.—The amount of stock purchased for £918 in the $3\frac{1}{2}$ per cents. at 90 = £$\frac{918 \times 100}{90}$ = £1020.

Again, if at 85, the amount of stock that could have been purchased = £$\frac{918 \times 100}{85}$ = £1080.

$\therefore$ the amount of additional stock = £1080 − £1020 = £60.

Ex. 7. What should be the price of the 5 per cent. stock when the rate of interest is 8 per cent.?

Sol.—An annual interest of £8 means an investment of £100 (cash).

$\therefore$ .....£5 ..... £$100 \times \frac{5}{8} =$ £$62\frac{1}{2}$.

That is, $62\frac{1}{2}$ ought to be the price of the 5 per cent. stock.

Caution.—The student is warned against supposing that the price of the 5 per cent. stock cannot but be $62\frac{1}{2}$ when the rate of interest is 8 per cent. The price of stocks depends, as has been explained above, on "an infinite variety of causes" of which the rate of interest is only one. Other things being equal, the price of the 5 per cents. is $62\frac{1}{2}$, interest being 8 per cent. per annum.

Ex. 8. How should I divide £608 of my money between the 3 per cents. at 80 and the 4 per cents. at 96, so that I may derive the same income from each?

Sol.—If I have £100 stock of the 3 per cents. I should have £75 stock of the 4 per cents. to derive the same income from each.

That is, if I invest £80 in the 3 per cents., I should invest (£96 × $\frac{3}{4}$ or) £72 in the 4 per cents.

Thus, If I have a sum of (£80 + £72 or) £152, I should invest £80 in the 3 per cents. and £72 in the 4 per cents.

But the sum that I have got to invest is £608, which is 4 times £152.

Hence I should invest £320 in the 3 per cents. and £288 in the 4 per cents.

Ex. 9. I hold £8,000 in the 3 per cents. which I sell out at 96; if the purchase-money is invested in the 5 per cents. at 28 premium, what change does it make in my income?

Sol.—By selling £8,000 stock at 96, I get 80 × £96 cash. Now by investing £128 cash in the second stock, I get an income of £5.

∴ by investing 80 × £96 cash, I get $80 \times £96 \times \frac{5}{128} = £300$ as income.

My income in the former case was at £3 for £100 stock; hence my total income in the former case was 80 × £3 or £240.

My income in the latter case is = £300.

Thus I derive an additional income of £60 now.

Ex. 10. I buy 20 shares of £6 when they are at 1 premium and sell them at 2 discount. Find my gain or loss.

Sol.—The sum for which I buy 20 shares = $20 \times £(6+1) = £140$.

The sum for which I sell the shares = $20 \times £(6-2) = £80$.

Thus I suffer a loss of £60.

Ex. 11. Which is more profitable—to buy an estate at 18 years' purchase or to invest in the 5 per cents. at 80?

Sol.—To buy an estate at 18 years' purchase means that we pay an amount equal to 18 times an year's rental. In other words, if the purchase money be Rs. 100, the annual rent will be $\frac{1}{18}$ Rs., *i.e.*, such an investment yields $\frac{100}{18}$ or $5\frac{5}{9}$ per cent. interest per annum, while the investment in 5 % at 80 yields $5 \times \frac{100}{80}$ or $6\frac{1}{4}$ per cent. interest per annum. Hence the latter investment is more profitable.

Exercise XV.

1. (*a*) Explain the terms *Stocks*, *Brokerage*, and the *Funds*.

(*b*) Explain the extract 'Consols were first quoted at $98\frac{3}{8}$, whence they improved to $98\frac{5}{8}$.'

2. What sum of money should be invested in the 5 per cents. at 98 to purchase £750 stock ?

Find the quantity of stock purchased by investing

3. £2,280 in the 3 per cents. at 77.

4. £3,048 in the 5 per cents. at $63\frac{3}{8}$, brokerage $\frac{1}{8}$ per cent.

5. £7,317 in the $3\frac{1}{2}$ per cents. at 10 discount.

6. Rs. 3,570 in the $2\frac{3}{4}$ per cents. at 75.

7. Rs. $3,533\frac{1}{2}$ in the 4 per cents. at $95\frac{3}{8}$, brokerage $\frac{1}{8}$ per cent.

8. If a stock-broker buys for me £2,300 stock, what sum does he receive, brokerage being $\frac{3}{16}$ per cent. ?

Find the sum of money got by selling

9. £575 in the $4\frac{3}{8}$ per cents. at $63\frac{7}{8}$, brokerage $\frac{1}{8}$ per cent.

10. £735 in the 6 per cents. at 4 discount.

11. Rs. 330 in the 3 per cents. at $95\frac{1}{8}$, brokerage $\frac{1}{8}$ per cent.

12. Rs. 480 in the 4 per cents. at $73\frac{7}{8}$, brokerage $\frac{1}{8}$ per cent.

13. £200 in the 4 per cents. at 93, (i) without brokerage, (ii) brokerage being $\frac{1}{8}$ per cent.

Find the yearly income arising from investing

14. £360 in the $5\frac{1}{4}$ per cents. at 84.

15. £1,305 in the $3\frac{1}{2}$ per cents. at $72\frac{3}{8}$, brokerage $\frac{1}{8}$ per cent.

16. Rs. $1,417\frac{1}{2}$ in the $3\frac{1}{2}$ per cents. at 81.

17. Rs. 1,734 in the $4\frac{1}{2}$ per cents. at 2 premium.

What sum of money must be invested in

18. The $3\frac{3}{4}$ per cents. at 80 to produce £60 ?

19. The $5\frac{1}{2}$ per cents. at $73\frac{1}{8}$ to produce £$71\frac{1}{2}$, brokerage $\frac{1}{4}$ per cent ?

20. The $3\frac{3}{4}$ per cents. at 5 discount to produce Rs. 645 ?

21. The $3\frac{5}{8}$ per cents. at 96 to produce £899 ?

22. The $3\frac{1}{2}$ per cents. at 84 to produce £70 ?

At what per cent. will a person receive interest who invests his capital in

23. The 5 per cents. at 5 premium ?

24. The $3\frac{3}{4}$ per cents. at 10 discount ?

25. The $3\frac{1}{2}$ per cents at $83\frac{7}{8}$, brokerage $\frac{1}{8}$ per cent. ?

26. The 3 per cents. at 78 ?

27. A person invested £6,940 in the 3 per cents. at $86\frac{5}{8}$ and sold out his stock when they have risen to $101\frac{1}{16}$. What was his gain, the brokerage in each transaction being $\frac{1}{8}$ per cent.?

28. A person transfers Rs. 5,000 stock from the 4 per cents. at 96 to the 3 per cents. at 80. How much stock does he hold in the latter ?

29. How much more stock would a person who invests £1,540 in the $5\frac{3}{4}$ per cents. at $87\frac{1}{2}$ have obtained, if he had waited till they had fallen to $82\frac{1}{2}$?

30. Which is the better investment :—investing in the 4 per cents. at 87 or in the 5 per cents., at 110 ? Check your result by drawing the graphs for the two stocks.

31. Represent the following fluctuations graphically :—

	Year.	Lowest Price.	Year.	Lowest Price.
Consols $2\frac{1}{2}$ %	1900	$96\frac{3}{4}$.	1905	$87\frac{1}{2}$.
	1901	91.	1906	$85\frac{9}{16}$.
	1902	$92\frac{1}{8}$.	1907	$80\frac{3}{4}$.
	1903	$86\frac{7}{8}$.	1908	$83\frac{1}{4}$.
	1904	85.	1909	$80\frac{1}{16}$.

32. Find by means of a graph what stocks at 120 and 125 will yield the same rate of interest as the 3 per cent. at 90.

33. Find by means of a graph how much money I must invest in 4% at 90 to produce an annual income of £26.

34 A person investing in the $4\frac{1}{2}$ per cents. receives 6 per cent. for his money ; what is the price of the stock ?

35. What is the rate of interest when the $3\frac{1}{2}$ per cents. are at 84 ?

36. I invest £3,990 in the $4\frac{1}{2}$ per cents. at 5 discount and £3,990 more when they have risen to 5 premium. If I sell the whole at par, how much do I gain or lose ?

37. It a person buys 36 shares of £6 5*s*. each when they are at 25 per cent discount and sells them at 5 per cent. premium, what is his gain ?

38. If a railway company open a line out of money raised by a 6 per cent, loan at 94 and realise 8 per cent. on actual outlay, find the net rate of profit.

39. A wishes to settle on B an annuity of £192 in the 3 per cents. when they are at $92\frac{3}{8}$. What sum of money must he invest, brokerage being $\frac{1}{8}$ per cent.?

40. I invest £7,560 in the 3 per cents. at $94\frac{1}{2}$ and when the funds fall to 90, I sell a quarter of my stock. When the funds are at $94\frac{3}{4}$, the remainder of the stock is sold. What is the alteration in my capital ?

41. A man invests £153 14*s*. 4*d*. in the $3\frac{1}{3}$ per cents. at 92 and £184 12*s*. $6\frac{1}{2}$*d*. in the $2\frac{1}{3}$ per cents. at 85. Show that he will receive the same income which he would have received had he invested the whole sum in the 3 per cents. at 90.

42. A man invests £1,404 16*s*. in the $2\frac{3}{4}$ per cents. at 99. How much must he invest in the $3\frac{1}{2}$ per cents. at 105 that the rate of interest on his whole investment may be 3 per cent. ?

***43.** How must I divide £5,476 16*s*. between the $2\frac{1}{4}$ per cents. at 37 and the $4\frac{1}{4}$ per cents. at 75, so that I may obtain the same income from each ?

***44.** A person who derives an income of £32 from the $4\frac{1}{2}$ per cents, sells his shares at 1 discount and by investing the proceeds in the 4 per cents. increases his income by £4. Find the price of the 4 per cent stock.

*45. A man possesses £22,400 of $2\frac{3}{4}$ per cent. consols which he sells at $99\frac{1}{2}$. He then invests one-half of the proceeds in 4 per cent. railway debenture stock at $133\frac{1}{2}$, lends £7,266 13*s.* 4*d.* on mortgage at $4\frac{1}{2}$ per cent. and loses the rest. What alteration is produced in his annual income ?

*46. A paper manufacturing company advertises for subscriptions for Rs. 1,20,000 5 % preference shares of Rs. 10 each, Rs. 1,00,000 ordinary shares. If the gross profits are Rs. 12,500 and if from this there be paid out for general expenses, reserve fund, &c., the sum of Rs. 3,025, find the rate of dividend paid to an ordinary shareholder and what dividend does he get on each share ?

*47. A Motor Car Company offers for subscription Rs. 14,00,000 Preference shares of Rs. 100 each, Rs. 16,00,000 ordinary shares of Rs. 100 each. The company estimates a profit of 4 as. per mile per day with runs of 120 miles per day for each of 100 cars which would run every day. It allows Rs. 45,000 for administrative expenses. Calculate the estimated profits ; what dividend can be declared on the ordinary shares if Rs. 1,500 be placed in the reserve fund ?

*48. A rice mill is started and a capital of Rs. 12,50,000 for the working of the mill is contributed by shareholders and at the end of one year a dividend of 3 per cent is declared. Shares were issued on the following conditions at the price of Rs. 100 per share

Payable on application	... 10%
On allotment	... 15%
10th June 1910	... 25
10th September 1910	... 25
10th December 1910	... 25

A man buys shares which yielded him Rs. 240 a year. What are his several instalments ?

*49. Electric Supply Company:—the quantity of electrical energy sold in a certain year amounted to 682,434 units producing a net revenue of Rs. 3-8-0 per unit. From this a sum of Rs. 10,000 is carried to the reserve and Rs. 6,525 is carried forward. The Directors propose dividend of 4 per cent. What was the share-capital ?

***50.** The following is an extract from the *Hindu* of a statement of affairs of the Bank of Madras :—

"Notice is hereby given that a dividend for the half-year ended 30th June 1911 of Rupees thirty per share being at the rate of 10 per cent. per annum and bonus of 1 per cent. on the capital stock of the Bank will be payable to the proprietors free of income-tax on the 13th instant until which date the transfer books will be closed."

Dr. **Profit and Loss Account.** **Cr.**

	RS.	A.	P.		RS.	A.	P.
To Dividend at 10 per cent. per annum on Capital Stock ...	3,00,000	0	0	By Balance ...	1,52,843	5	3
To Bonus at 1% ...	60,000	0	0	By Net profit for Half-year ended...	7,25,104	5	9
To Reserve Fund ...	52,00,000	0	0	By Reserve Fund on 31st December 1910 ...	49,50,000	0	0
To Premises ...	25,000	0	0				
To Balance carried forward ...	2,42,947	11	0				
Total ...	58,27,947	11	0	Total ...	58,27,947	11	0

Find (1) the capital stock of the bank ;

(2) the nominal value of each share ;

(3) the sum placed in the reserve fund.

R.B.A.N.M.'S HIGHER SECONDARY SCHOOL
BANGALORE-1.

CHAPTER XVI.

MISCELLANEOUS.

TIME AND WORK.

Some advanced exercises on time and work, time and distance, profit and loss, &c., will be presented in this chapter suitable for a High School course.

Ex. 1. A's rate of working : B's :: 3 : 4 and B's : C's :: 1 : 2. How long will it take C to do what A would do in 6 days?

Let the rate of working of A, B, C be denoted by a, b, c. respectively.

$a : b :: 3 : 4$, *i.e.*, $4a = 3b$;

$b : c :: 1 : 2$, *i.e.*, $c = 2b$.

$$\therefore \frac{4a}{3} = b = \frac{c}{2}. \quad \therefore a = \frac{c}{2} \times \frac{3}{4} = \frac{3}{8}c.$$

i.e., A works at $\frac{3}{8}$ of C's rate.

$\therefore$ what A can do in 6 days, = what will be done in 6 days working at $\frac{3}{8}$ of C's rate.

$\therefore$ the time required to do that work working at C's rate

$$= \frac{6 \times 3}{8} = 2\tfrac{1}{4} \text{ days.}$$

Ex. 2. A pipe A which can fill a cistern in 3 hours is open at 8 A. M. The pipe B which can empty it in $\frac{3}{4}$ hour is turned on at 9-30 A.M. When will the cistern be empty?

In the interval between 8 A M and 9-30 A.M., *i.e.*, in $1\frac{1}{2}$ hours the first pipe fills $\frac{1}{2}$ the cistern. After 9-30 both pipes are working. In 1 hour the second pipe can empty $\frac{4}{3}$ cistern and the first pipe can fill $\frac{1}{3}$ cistern.

$\therefore$ 1 cistern is emptied, when both the pipes are open, in 1 hour.

$\therefore$ $\frac{1}{2}$ cistern ..... $\frac{1}{2}$ hour,

i.e, at 10 A.M. the cistern is empty.

Graphical Solution. Let one small division on the x axis (Fig. 148) represent 5 minutes and 12 small divisions on the y axis the whole cistern The origin is 8 A M. Plot the point P, (11 A M., 1 cistern). Join OP. This is the

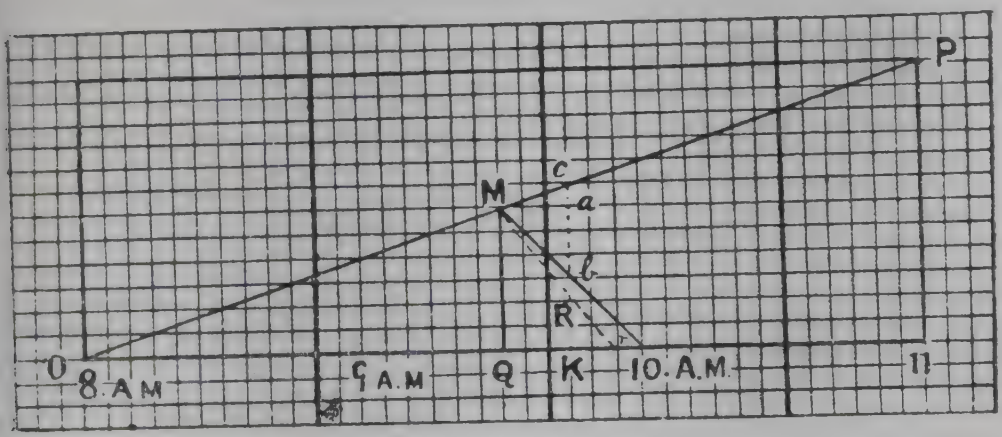


FIG. 148.

graph of the filling pipe. At 9-30 A.M., draw the ordinate QM meeting OP at M, QM represents the quantity of water collected in the cistern at 9-30 A.M. After 9-30 both pipes are working. From M take a move 3 small divisions to the right and then step 4 small divisions below so as to come to the point R. MR represents the graph of the emptying pipe. (The line is shown dotted in the figure). We have to consider the combined effect of both the pipes. Now in the time represented by QK after 9-30, ac represents the water brought in by the first pipe, aR that removed by the second pipe. Hence with your dividers cut off $ab = a\text{R} - ac$. Join Mb. Then Mb is the graph of both the pipes. Mb

produced meets the axis at 10 which tells us that there is no water in the cistern at 10 o'clock.

Note. If the first pipe is closed at 9-30 and the second alone is working, then we should consider only the graph of the second pipe, *viz.*, MR which meets the x axis at 9-52$\frac{1}{2}$. The cistern will in this case be emptied at 9-52$\frac{1}{2}$ A.M.

Ex. 3. A cistern 35 inches deep is connected to two pipes one of which A fills the cistern and the other B empties it. At 4 A.M., when there are 10 inches of water in the cistern, A is turned on. At 5 A.M., there are 12$\frac{1}{2}$ inches; at 6 A.M. 15 inches; at 7 A.M. 17$\frac{1}{2}$ inches; at 8 A.M. 16 inches; at 9 A.M. 14$\frac{1}{2}$ inches; and at 10 A.M. 13 inches when it is noticed that the pipe B had been accidentally opened; it is required to find when the pipe B got opened.

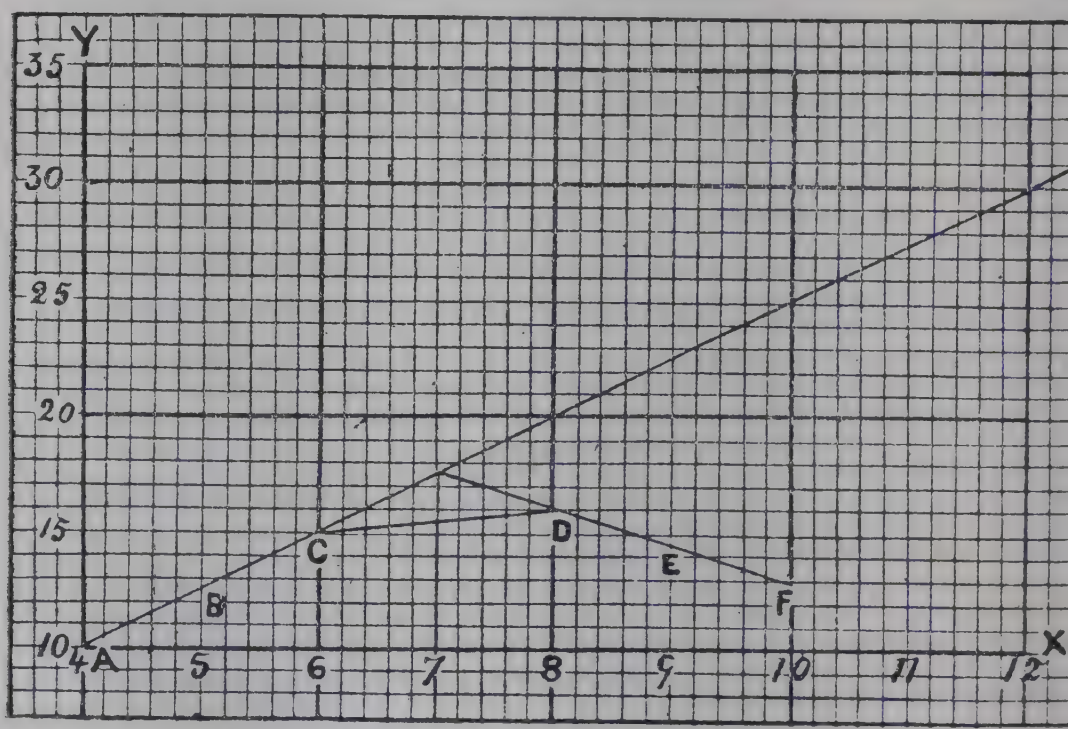


FIG. 149.

On the x axis (Fig. 149) set off 5 divisions to represent 1 hour and on the y axis 1 division to represent one inch. Plot the various points given in the question, taking (4 A.M, 10) as the origin. Join the points successively by straight lines. ABCDEF is the graph formed. Now, since A, B C are found to be in one straight line, it is clear that ABC is the graph of the pipe A and for a similar reason DEF must be the graph of A and B running together and that the part CD, by its upward slope, represents that the pipe A must have been running the whole period of 2 hours from 6 to 8 and the pipe B a part of that time. If therefore we continue ABC and DEF upwards till they meet, then the abscissa of the point of intersection will give the time at which B began to run ; *viz.*, at 7 A.M. from the graph.

But this is not all that we may find out from this diagram regarding the pipes and cistern.

The graph ABC of the pipe A will be found to meet, if produced, the parallel to the x axis through 35 on the y axis, at a point whose abscissa is 2 P.M.

Also DEF is the graph of pipes A and B. Through 35 on the y axis draw a line parallel to DEF] cutting the x axis produced. Say, at P. Then the abscissa of P gives the time of emptying the whole cistern, when both the pipes are open, *viz.*, $23\frac{1}{3}$ hrs.

[For want of space the line is not shown in the figure].

Further as the level of water falls $1\frac{1}{2}$ inches per hour, when both the pipes are open, although A fills in $2\frac{1}{2}$ inches, $\therefore$ B must run out 4 inches per hour. Hence by plotting 1, 4 and drawing the graph of B, we would find the cistern is emptied, when B alone is working, in $8\frac{3}{4}$ hours.

Ex. 3 There is a well full of water. Two pumps, which are calculated to empty the well in 25 and 20 minutes respectively, are set working. But the well is

constantly supplied by a spring which can refill it in half an hour. The two pumps work together for 15 minutes, when that of larger capacity ceases work for 10 minutes. The 2 pumps again work together, until the well is empty How long will each pump have been employed?

Graphical Solution. Let OY represent the depth

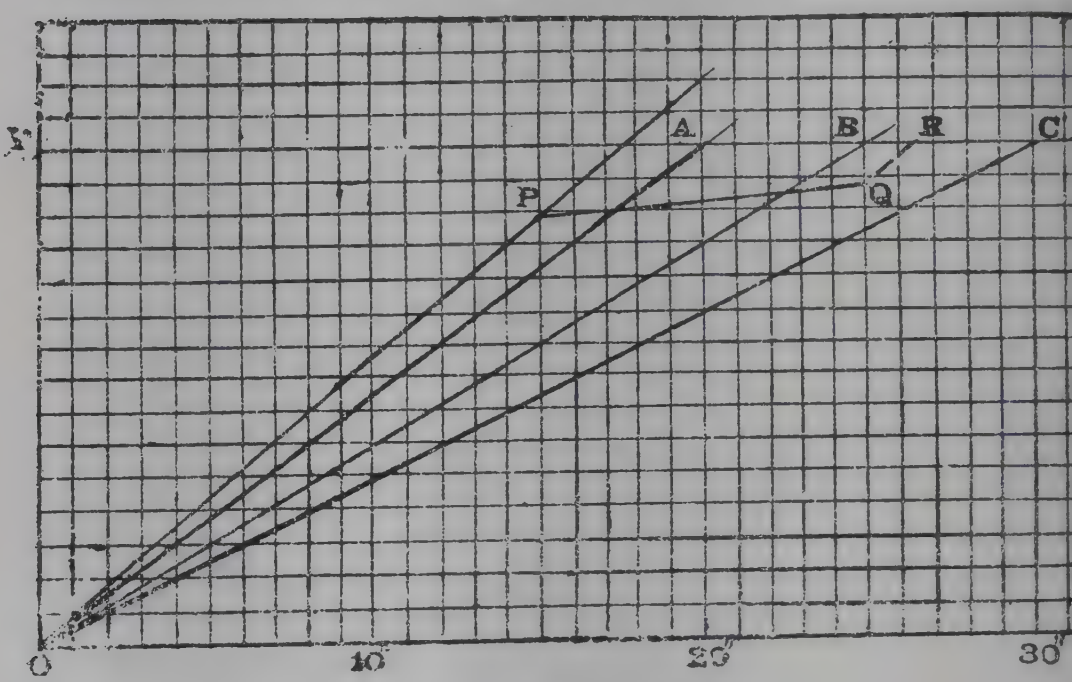


FIG. 150.

of water in the well Let time be represented on the other axis, each division representing a minute. [See Fig. 150.]

Since one pipe empties the well in 20 minutes, OA is the graph representing its work. the ordinate of any point on OA representing the depth of water emptied by the pipe in the time represented by the corresponding abscissa.

Similarly, OB and OC will be the graphs respectively of the second emptying pipe and the spring; in the case of the spring, the ordinate represents the depth of water *filled* in by it.

When the pipes and the spring work together, the effect will be represented by the graph OP, which is obtained thus : measure the ordinates of points on OA, OB and OC corresponding to any abscissa, say 10^m ; from the sum of the first two ordinates, subtract the third ; the result will be the ordinate showing the depth of water emptied in 10^m.

The pipes work for 15^m. The quantity of water emptied in 15^m is given by the ordinate of P corresponding to 15^m. Then the 1st pipe ceases work. The graph of the net work done by the 2nd pipe and the spring together is PQ ; (which is obtained in the same way as OP was obtained ; only, the first pipe is now ignored). This state of affairs continues for 10^m. At the end of this time, *i.e* , 25 minutes from the beginning, the quantity of water emptied is given by the ordinate of Q corresponding to 25^m. Again, through Q, draw QR parallel to OP, as all three work together again. The abscissa of R, (at which point QR cuts the horizontal through Y) gives the time when the well is empty ; which is about $26\frac{1}{2}$ minutes.

(The Algebraical Solution is left as an exercise to the student.)

Exercise XVI (a).

1. 20 men and 15 boys can do a piece of work in 16 days. If each man takes 5 hours for what each boy requires $7\frac{1}{2}$ hours to do, find the time in which (*a*) 1 man, (*b*) 1 boy will be able to do the same work.

2. A and B can do a piece of work in 15 and 18 days respectively ; they work together at it for 3 days when B leaves, 3 days after A is joined by C and they finish it together in 4 days. In what time can C do the piece of work ? Illustrate your answer graphically.

3. 13 men and 16 boys can do a piece of work in $7\frac{1}{2}$ days ; they work at it till they complete half of it ; then two of the men leave and 2 more boys come ; how long will the work be on hand, if a boy does half as much work as a man ?

4. A can do a work in 10 days, B in 20 and C in 30 days. They work together for a number of days when A leaves; B and C work for 3 days more when B leaves. If the whole work is finished in 12 days more, what part of the work is done by B?

5. A is twice as good a workman as B and thrice as good as C. Working together for 10 days they can finish the work. They all begin together. But after working for 3 days A leaves off. After 5 days more B also leaves off. In how many days more will C finish the work?

6. A and B could separately reap a field in 6 and 5 days respectively. If they work on alternate days at it A beginning, which of them will finish the work and what fraction of a day's work will he do on this last day? Illustrate graphically.

7. A alone can do a piece of work in 15 days and B in 12 days. A begins alone and after a certain interval B joins him and the work is finished in $7\frac{1}{2}$ days. How long were A and B working together? Solve this graphically.

8. A certain number of men and women can do a piece of work in 20 days. If there had been twice as many women more the work would have been finished in 12 days. In how many days can the women do the work alone?

9. A certain number of men do a piece of work in 10 days. If there were 3 men more, it could be finished in 2 days less. Illustrate graphically.

10. On a piece of work 3 men and 6 boys are employed who do half of it in 6 days. After this, two more men are put on and $\frac{1}{3}$ more is done in 3 days. How many more men must be put on that the whole work may be completed in 1 day more?

11. If 20 oxen can consume the grass of a meadow in which the grass grows uniformly in 96 days, and 30 oxen can consume the same in 60 days, how many oxen will eat in 24 days?

12. Supposing that a tank has a regular and constant supply of water and that 12 men can empty it in $7\frac{1}{2}$ minutes and 7 men can empty it in 16 minutes, how many men would empty it in 50 minutes?

13. A cistern can be filled by a pipe A in 25 minutes and emptied by a pipe B in 40 minutes. They are opened simultaneously,

but A is closed for 1 min. at the end of every 5 minutes and B for 2 minutes at the end of every 4 min. In what time will the cistern be filled? Illustrate graphically.

14. A besieged garrison has a supply of water for 50 days. Owing to a leak however, in the bottom of the reservoir 5 gallons waste away every day and then the supply suffices for 10 days less; find for how many days the supply would suffice if 20 gallons leak off every day.

15. 30 men begin a work which they can finish in 50 days. If the men drop off by 4 at the end of every 12 days, how long will it be before the work is finished? Illustrate graphically.

Time and Distance.

Ex. 1. The following are extracts from a Railway time-table, the distances being measured from London :—

Passenger train.			*Express.*	
London		... 1-10	Bath ...	... 2-35
Reading (36 miles)	arr.	... 1-55	London	... 4-25
	dep.	... 2-2		
Swindon ($77\frac{1}{4}$ miles)	arr.	... 2-53		
Bath (107 miles)	arr.	... 3-39		

Assuming that the fast train travels uniformly and passes the stopping train as the latter leaves Swindon, find how long the latter stops at Swindon.

Sol.—The particulars given above on the left side relate to an ordinary train stopping at the intermediate stations; while the particulars on the right relate to an express train.

In fig. 151, distance is represented on one axis, 5 miles being represented by a single division. Time is represented on the other axis, each division representing 5^m. The point (1-55, Reading) is joined to (1-10, London) and the line is the graph of the motion of the ordinary train between London and Reading. Between 1-55 and 2-2, the train was stationary and its graph is therefore parallel to the time-axis, the distance-ordinate being the same.

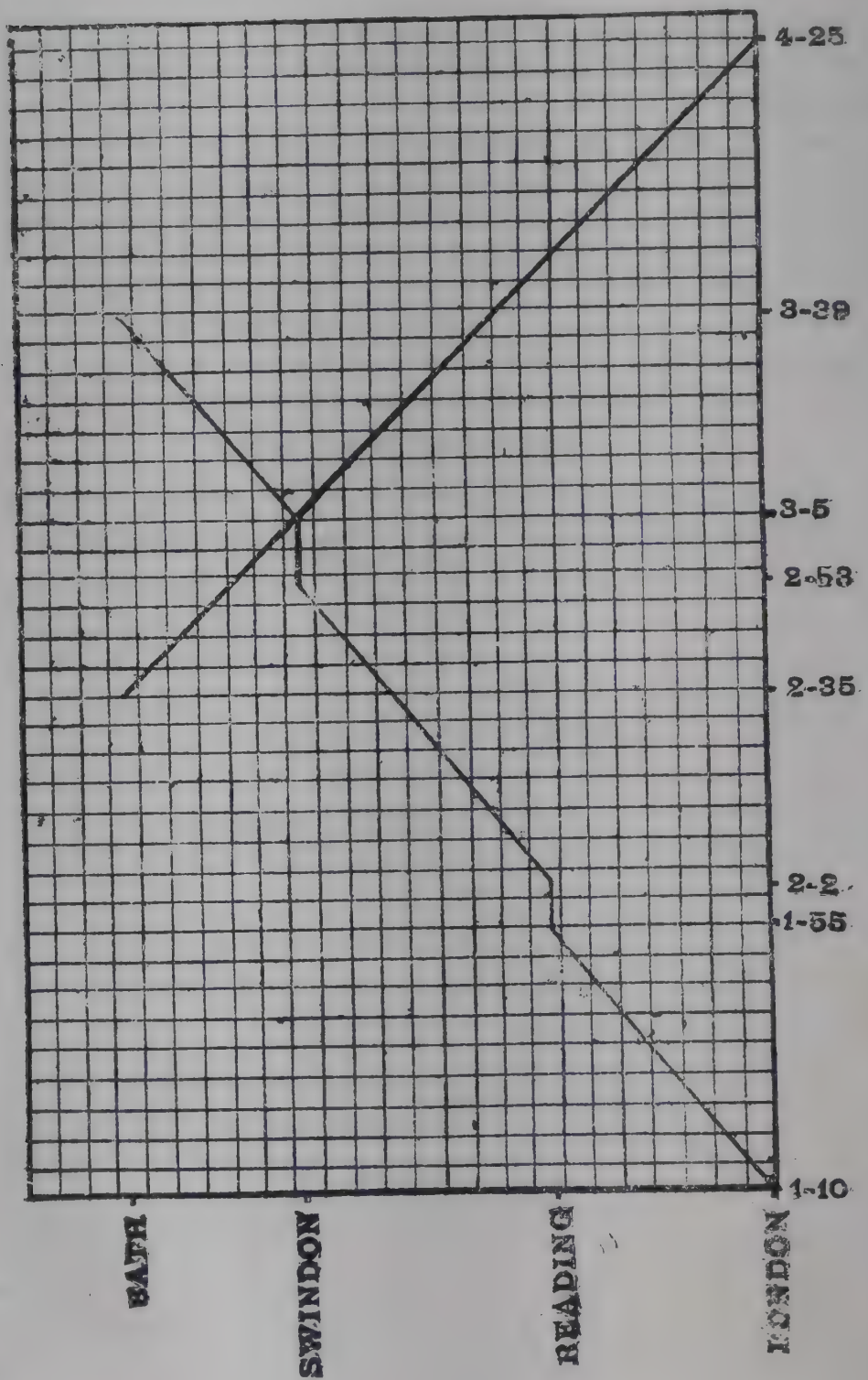


FIG. 151.

Next points (2·2, Reading) and (2·53, Swindon) are joined, and you get the graph of the motion between those two places.

The graph of the motion of the express train is given by joining (2·35, Bath) to (4·25, London). The abscissa of the point where this graph cuts the parallel through Swindon will give the time when the express passes Swindon; which is, 3·5. So, the ordinary train halts at Swindon for 12 minutes.

Ex. 2. Two trains P and Q running on parallel lines enter a tunnel 1 mile long at opposite ends at the same time. They meet in 45^s, and P has then travelled in the tunnel 220 yds. more than Q. In $4\frac{1}{2}^s$ they cross each other, and $34\frac{1}{2}^s$ later P has completely passed through the tunnel. Find the lengths and rates of the trains.

Sol.—Let the time be represented on OX, (Fig. 152) each division representing 3 seconds. Let the distance travelled be represented on OY, at 55 yds. per division. As the train P leaves O, Q leaves the other end of the tunnel, namely, point (0, 1760 yds.) In 45 seconds they meet and P has travelled 990 yds., while Q has travelled 770 yds. in the 45^s. So the graph of the motion of P is given by joining O to the point (45^s, 990 yds.) and is OA. Again the graph of Q is obtained by joining (0, 1760 yds.) to (45^s, 990 yds.)

In $4\frac{1}{2}$ seconds they pass each other. And at the instant when they have just passed each other, the distance AB, between the engines of the two trains is the sum of the lengths of the trains. Again $34\frac{1}{2}^s$ later, P has completely passed through the tunnel, and at that instant, the whole length of the train P will be just outside the tunnel. In the figure, CD shows the distance described beyond the tunnel, and $\therefore$ represents the length of P.

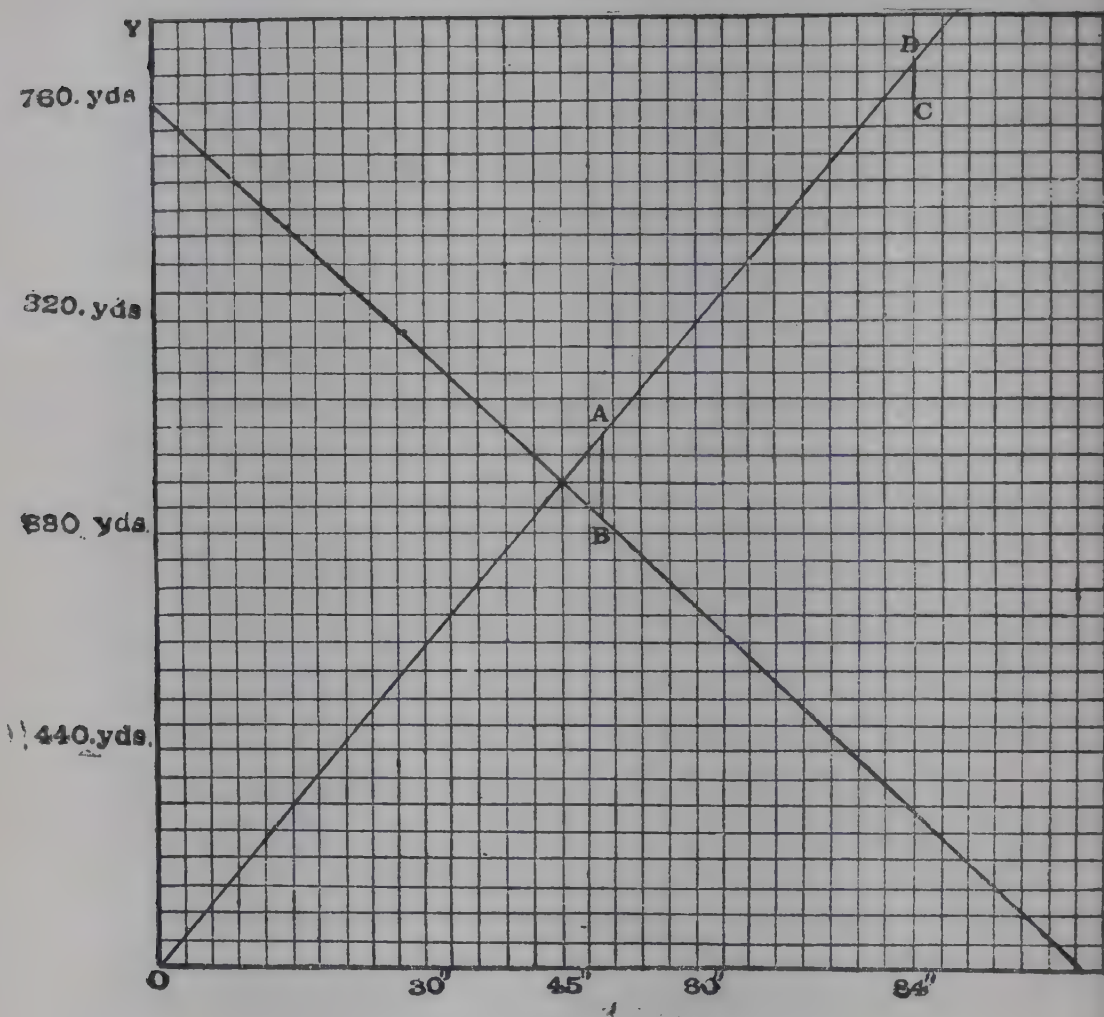


FIG. 152.

The sum of the lengths of the trains is thus given by AB and is equal to about 170 yds. from figure; and CD (the length of P) is about 83 yds. [Working by the ordinary method of Arithmetic, the lengths of the trains will be seen to be 88 yds. each].

The rates of P and Q are 990 yds. per 45^s and 770 yds. per 45^s respectively; or 45 miles and 35 miles per hour respectively.

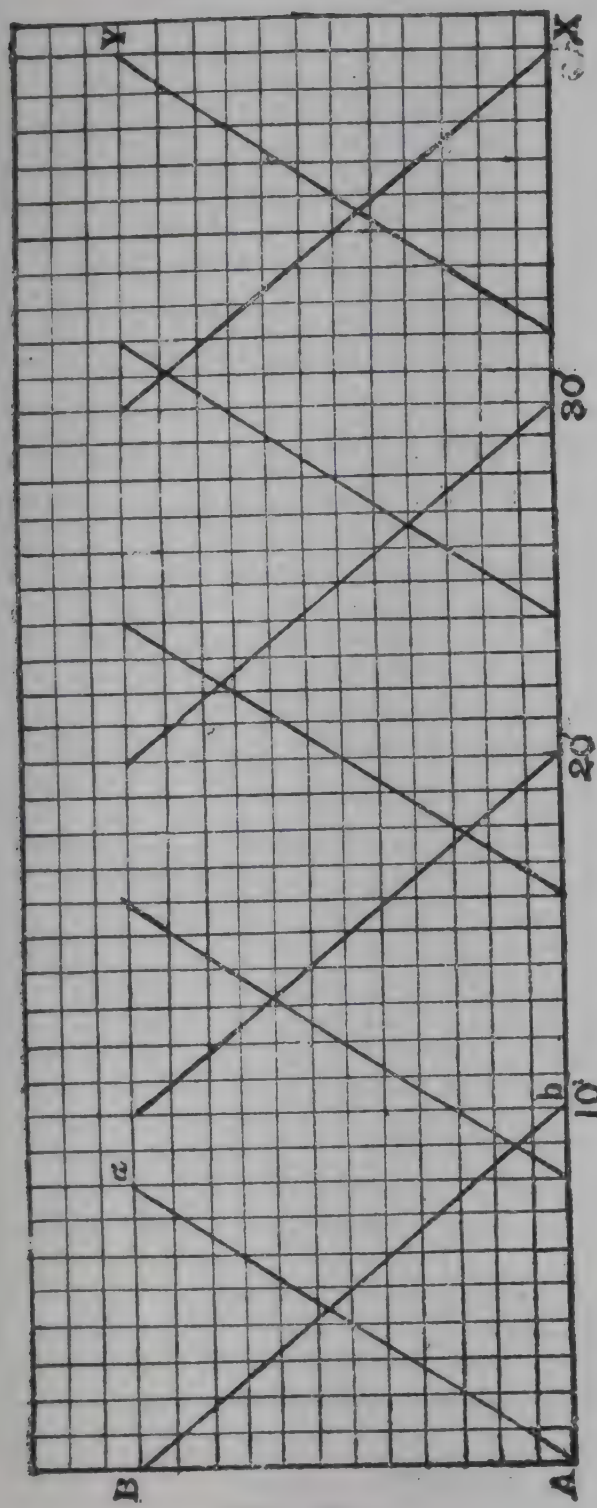


FIG. 153.

Ex. 3. Two persons L and M run in opposite directions round a circular race-course. They can finish the course in 8 and 10 minutes respectively. (1) When do they meet for the 4th time? (2) When will they meet at the starting point?

Sol.—Let AB (Fig. 153) represent the length of the course, and let each division along AX represent a minute. Then A*a* will be the graph of the motion of L. Since the course is circular, the end of the course is the same as the starting point. So, the graph of the motion of L during the second eight minutes, is the

parallel to Aa through the point $(8^m, 0)$. Thus, Aa and its parallels represent the motion of L.

Similarly Bb and its parallels represent the motion of M.

(1) They meet for the 4th time at a point whose abscissa is about $17\frac{3}{4}$ minutes. [The exact time is $17\frac{7}{9}$ minutes.]

(2) When L is at Y, M arrives at X. Since X and Y represent the same point on the circular race-course, the two persons are found to arrive at the starting point after 40 minutes.

NOTE.—From the same figure may be found when the two persons are a given distance apart; a quarter of

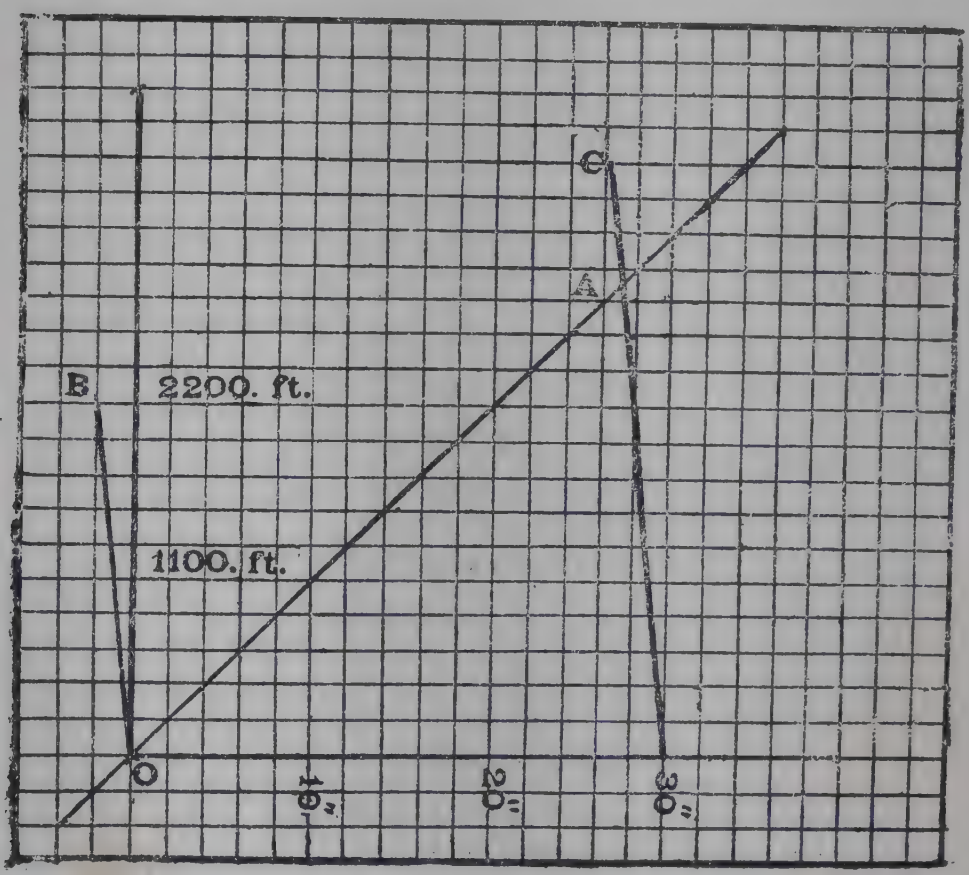


FIG. 154.

the circumference for instance, for the 4th time. Since the race-course is circular, a distance of $\frac{3}{4}$ of the circumference will also count as a distance of a quarter. To find the required time here, we must draw all parallels to AB from which lengths of 9 divisions and 3 divisions are intercepted by the graphs of motions of L and M. [12 divisions represent the length of the race-course.] The 4th such intercept will give the required time which is about $7\frac{3}{4}$ minutes. [The exact answer is $7\frac{7}{9}$ minutes.]

Ex. 4. Guns are fired from a place at intervals of 30^s. A train, travelling at the rate of 75 miles an hour, approaches the place. At what intervals will a passenger in the train hear the reports? Sound travels at 1100 ft. per second.

Sol.—Let each division on the horizontal through O represent 2^s and each division on the vertical through O represent 220 ft Let us suppose the first report was heard at O. The graph of the motion of the train is OA, (for 75 miles an hour is the same as a rate of 110 ft. per second) and the graph of that of the sound is BO.

The line through 30^s parallel to BO, *viz.* CA will be the graph of the sound corresponding to the second report of the gun. If the train were stationary its graph would be the line joining O and 30^s, no distance being travelled by the train; and the second report would be heard 30^s after the first. But as it is, the two graphs OA and CA meet at A, whose abscissa gives the time to be a little over 27^s.

Note.—In the above example, as well as in similar examples where graphical solutions are given, the student is warned against supposing OA to be the *direction* of the train. The actual motion is in the direction of the axis representing distance.

Ex. 5. A train 44 yards long overtook a person walking along the line at 6 miles an hour and passed him in 5 seconds. 20 minutes afterwards it overtook another person and passed him in $4\frac{1}{2}$ seconds. When will the first man overtake the second?

Sol. In 5 sec. the train gains 44 yds. over the first man.

$\therefore$ in 1 hr. or 3600 sec. the distance gained $= \frac{44 \times 3600}{5 \times 1760}$ or 18 miles.

$\therefore$ the rate of the train is 18 + 6 or 24 miles an hour.

Again, in $4\frac{1}{2}$ sec. the train gains 44 yds. over the second man.

..... 1 hr..... $\frac{44 \times 3600 \times 2}{9 \times 1760}$ or 20 miles.....

$\therefore$ the rate of the second man $= 24 - 20 = 4$ miles an hour.

Now, the train overtakes the second man 20 min. after it passes the first man.

$\therefore$ the distance between the first man and the train at the time of its overtaking the second man is the distance gained by it over the first man in 20 min., *i.e.*, $18 \times \frac{20}{60}$ or 6 miles.

$\therefore$ the distance between the two men is $6\frac{44}{1760}$ miles;

but the first man gains 6—4 or 2 miles over the second man in 1 hr.

$\therefore$ the time required $= \frac{6\frac{1}{40}}{2}$ hrs. $= 3$ hrs. 0 m. 45 secs. after the train overtakes the second man.

Ex. 6. A and B start at the same time from P to Q. A, who walks at 6 miles an hour, goes to Q, returns to S (a distance of 4 miles) and turns again and reaches Q

8 hrs. after starting, at the same time as B, who walks uniformly from P to Q. Find B's rate of walking and the distance from P to Q.

Sol.—In 8 hours A walks 8×6 or 48 miles, which is equal to the distance from P to Q + twice the distance from Q to S.

$\therefore$ the distance from P to Q $= 48 - 2 \times 4 = 40$ miles.

$\therefore$ B's rate of walking $= \frac{40}{8} = 5$ miles an hour.

Ex. 7. A railway train having left a terminus at noon is overtaken at 6 P.M. by another train which left the same terminus at 1 P.M. If the former train had been 10 miles further on the road when the latter started, it would not have been overtaken till 8 P.M. Find the rates of the trains.

Sol.—In 5 hrs. the second train gains the distance travelled by the first train in 1 hr.

In 7 hrs. it gains the same distance + 10 miles.

$\therefore$ in 2 hrs. the second train gains 10 miles.

$\therefore$ in 1 hr.... ..... $\frac{10}{2}$ or 5 miles.

$\therefore$ in 5 hrs..... 5×5 or 25 miles.

$\therefore$ the rate of the first train $= 25$ miles an hour

.....the second....$= 30$.....

Ex. 8. A and B start at the same time from P and Q (56 miles distant) to Q and P at 4 and 3 miles an hour respectively, meet at R, reach Q and P, return immediately and meet again at S. Find the distance from R to S.

Sol.—When A and B meet for the *first* time, they have together walked the whole distance PQ (56 miles); and when

they meet for the *second* time they have together walked *three* times the distance or 3×56 or 168 miles.

But in the time they together walk 7 miles A walks 4 miles.

$\therefore$ .....56 miles.....32 miles.

$\therefore$ PR = 32 miles.

When they have together described three times the distance, B has described QP + PS. But in the time they together walk 7 miles B walks 3 miles.

$$\therefore \quad QP + PS = \frac{168 \times 3}{7} = 72 \text{ miles.}$$

$\therefore$ PS = 72—56 or 16 miles.

$\therefore$ RS = 32—16 or 16 miles.

Exercise XVI (b).

1. A and B start at the same time from X to Y walking at 7 and 4 miles an hour respectively. A reaches Y, returns immediately and meets B at Z 5 miles from Y. Find the distance from X to Y.

2. A detective sets out in pursuit of a thief who has a start of 34 miles. At the end of 4 days he finds that the thief travels 8 miles a day faster than he does and so trebles his speed and overtakes him in 3 days. Find their rates of travelling.

3. A and B walk round a circular lake 48 miles in circumference starting from the same point simultaneously and walking at 4 and 5 miles an hour. (1) When will they meet.? (2) When will they be opposite each other for the first time? (3) When will they be quarter of a circle apart for the third time?

4. In a mile race A can beat B by 20 yds. or 3 minutes. Find A's time over the course.

5. C can beat B by 40 yds. in a 200 yds. race and can beat A by 15 yds. in a 135 yds. race. By how many yds. can A beat B in an 80 yds. race? Illustrate graphically.

6. A can give B 50 yds. start in a mile race and beat him in $5\frac{1}{3}$ minutes by 5 yds. At the same rate A can give C 10 seconds start and make a dead heat of the race. How many yds. start must be given to B that he may win the mile race by 4 yards against C ?

7. A race course is half-a-mile long. A and B run a race and A wins by 10 yds. ; C and D run over the same course and C wins by 30 yds. , B and D run over it and B wins by 20 yds. If A and C run over it, determine who wins and by how much ? Illustrate graphically.

8. A train 88 yds. in length overtook a person walking along the line at $5\frac{2}{3}$ miles an hour and passed him in 12 seconds. Twenty minutes later, the train overtook another person and passed him in 9 seconds. When will the latter be overtaken by the former ?

9. A person walking at 3 miles an hour met a train 88 yds. long which passed him in 6 seconds. A rider going in the same direction met the same train an hour later, which passed him in 5 seconds. When will the latter person overtake the former ?

10. A and B leave P for Q simultaneously. A who walks at 4 miles an hour walks straight and reaches Q in 5 hours. But B goes to S, returns to T (a distance of 6 miles) turns again and reaches Q with A. Find B's rate of walking.

11. Two trains start simultaneously from T and M at 7-30 A.M. and meet each other at S at 10 A.M. If T is 150 miles from M and if the rates of the trains are as 3 : 5, find (i) the distance from S to M and (ii) the rates of the two trains.

12. A goes from P to Q and finds that if his rate per hour had been half-a-mile faster, he would have performed his journey in $\frac{4}{5}$ of the time, but if it had been half-a-mile slower, he would have taken $1\frac{1}{2}$ hours more to do it. Find his rate and the distance from P to Q.

13. A and B walking in opposite directions start at the same time from P and Q respectively and meet in $3\frac{1}{3}$ hours. If A reaches Q in $2\frac{2}{3}$ hours more, compare their rates. If B's rate

be 1 mile less than A's per hour, find the distance from P to Q. Illustrate graphically.

14. A leaves T at 3 A.M. and reaches M at 4 A.M. while B who leaves M at 3-20 A.M. meets A at 3-45 A.M. When will B reach T. Iliustrate graphically.

15. A and B start at the same time from P and Q (distance 48 miles) to Q and P at 3 and 4 miles an hour respectively, meet at Y, reach Q and P respectively, return immediately and meet at Z. Find the distance from Y to Z.

16. A and B run a mile; at the first heat A gives B a start of 44 yds. and beats him by 57 seconds. At the second heat he gives him a start of 1¼ ~~mile~~ and is beaten by 88 yds. In what time can A and B run a mile separately?

17. Two clocks one of which gains 2 min. and the other loses 1 min. per day, are set right at noon. What is the time by the first clock when the second indicates noon, a week afterwards?

18. Two clocks strike together on Monday morning; on Tuesday morning one is 8 minutes to 12 when the other strikes 12; how much must the first be put on that they may both strike 8 together in the evening?

19. Two clocks one of which gains 1 minute and the other loses 5 minutes in 16 hours are set right on Monday morning, 25th February 1897. Find when they show the same time of day again together and what o'clock it is.

20. A and B start at the same time from P and Q (distance 54 miles) to Q and P at 5 and 7 miles an hour respectively, meet at Y, reach Q and P, return immediately and meet at Z. Find the distance from Y to Z. If the distance YZ = 20 miles, find the distance from P to Q.

Profit and Loss.

Ex. 1. A reduction of 20 per cent. in the price of sugar would enable a purchaser to obtain 3 seers more for a rupee. Find the original price of sugar per seer.

Sol.—The reduced price of sugar $= \frac{100 - 20}{100}$ or $\frac{4}{5}$ of the original price.

$\therefore$ the quantity that can be bought for the same sum $= \frac{5}{4}$ of the original quantity.

$\therefore$ $\frac{1}{4}$ of the quantity that can be bought for Re. 1 at the original price $= 3$ seers.

$\therefore$ the quantity that could be bought for Re. $1 = 12$ seers.

$\therefore$ the price of 1 seer $= 1\frac{6}{12}$ as. $= 1$ a. 4 p.

Ex. 2. If a merchant makes a profit of 15 per cent. in selling an article at Rs. 4-12-8, at what price must he sell to make a profit of 18 per cent.?

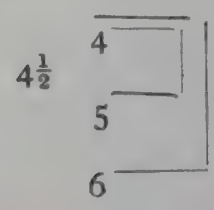
When the gain is 15 %, if the sale price be 115, the cost price would be 100; and $\therefore$ to gain 18 % the sale price should be 118.

Thus if the sale price in the first case be 115, the sale price in the second case must be 118.

$\therefore$ if the sale price be Rs. 4-12-8 the sale price in the second case must be Rs. 4-12-8 $\times \frac{118}{115}$ = Rs. 4-14-8.

Ex 3. How must a grocer mix sugar at 4 as., 5 as., and 6 as. per viss to obtain a mixture worth $4\frac{1}{2}$ as. per viss?

Sol.—Arrange the prices under one another in order, and place the given average price to the left of them as shown below.



$\frac{1}{2}$ viss + $1\frac{1}{2}$ visses.

$\frac{1}{2}$ viss.

$\frac{1}{2}$ viss.

Now, mix the sugars of the first sort and the second sort so as to get a mixture of the average price; that is, take

$\frac{1}{2}$ viss of the first sort and $\frac{1}{2}$ viss of the second sort. [For method, see Example 3, page 100.]

Again, mix the sugars of the first sort and the third sort so as to obtain a mixture of the average price; that is, take $1\frac{1}{2}$ visses of the first sort and $\frac{1}{2}$ viss of the third sort. [For method, see Example 3, page 100.]

A mixture of these mixtures will give us the required mixture of all the three; that is, 2 visses of the first sort, $\frac{1}{2}$ viss of the second sort, and $\frac{1}{2}$ viss of the third sort. Hence the ratio required is 4 : 1 : 1. *Ans.*

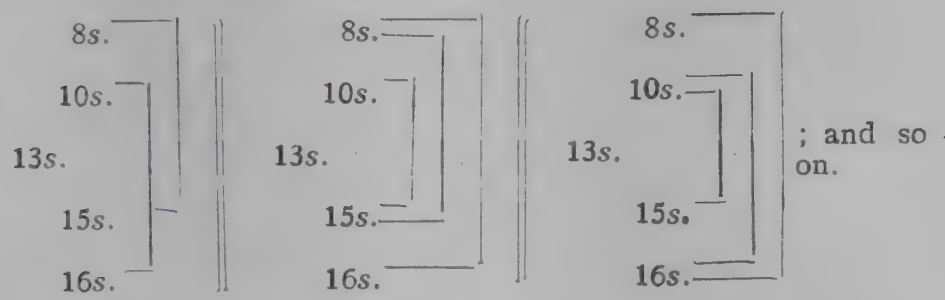
Ex. 4. How must a vintner mix four sorts of wine at 8*s.*, 10*s.*, 15*s.* and 16*s.* per gallon so as to form a mixture worth 13*s.* a gallon?

Sol.—

8*s.*		3 gals.	Write the prices under one
13*s.*	10*s.*	2 gals.	another in order, and place
	15*s.*	3 gals.	the given average price to the
	16*s.*	5 gals.	left of them.

Here there are two prices over the average price and two under it. We now "link" these prices together two and two, so that a price that is greater than the proposed average price shall be linked to one that is less. Now linking the first and fourth, we find that the grocer should take 3 gallons of the first and 5 gallons of the fourth so that there may be got a mixture of the given average price. Similarly linking the second and the third, we find that he should take 2 gallons of the second and 3 gallons of the third. Hence one of the ways in which the wines may be mixed is in the ratio of 3 : 2 : 3 : 5.

Note.—Questions of this kind admit of *a great number of solutions*, according to the manner of linking the prices. Thus, the prices in the question solved above, may also be linked in the following ways :—



We shall illustrate the same fact further, by a graphical solution of Example 3 above.

Let each division on the axis OX (Fig. 155) represent a palam, and each division on OY, 2 pies. Then OA, OB and OC will be the graphs of the prices of the three kinds, while OR will be the graph of the price of the mixture.

To see what quantity of each kind enters into the mixture, through R draw RP parallel to OA. [See solution to Example 5, page 100]. Take *any* point P in RP and through P draw PQ parallel to OC meeting OB in Q. Through P and Q draw PKM and QS perpendiculars to OX, and PN and QKT, perpendiculars to CX. Now the sum of the ordinates SQ, KP and NR = XR, and is the sum of the prices of quantities, represented by OS, QK and PN, of the second, third and the first kind. [For PQ being parallel to OC is the graph of the third kind of sugar, and PR, parallel to CA, is the graph of the first kind].

Therefore the three sorts of sugar will be mixed in the ratio of MX : OS : SM, *i.e.*, in the ratio of 24 : 12 : 4; or 6 : 3 : 1.

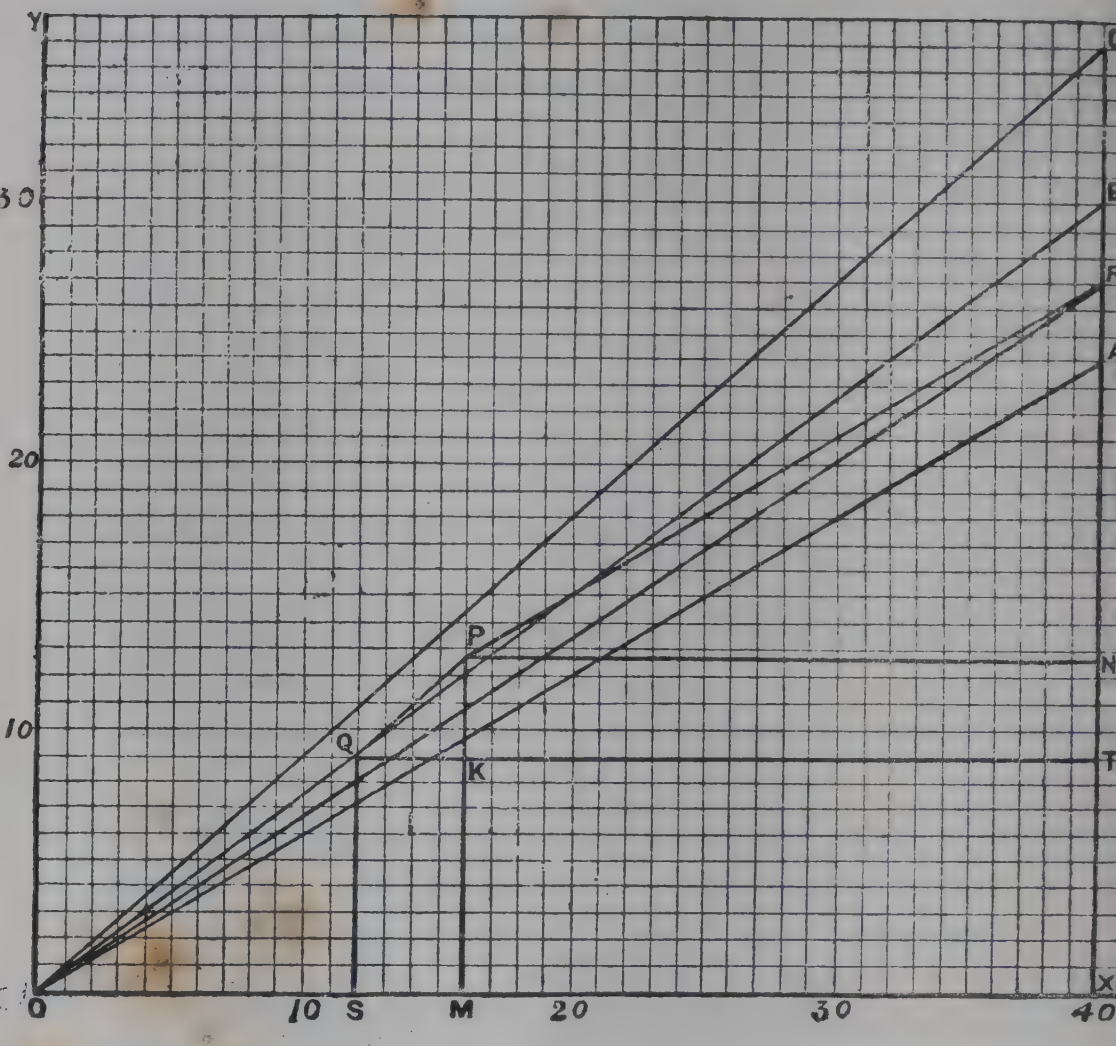


FIG. 155.

Now, we took *any* point P in RP for which we obtained the answer 6 : 3 : 1. But we may take any other point in RP, and repeating the construction, we shall obtain a different answer for the ratio of the ingredients, which will also be correct. Thus, there are as many solutions to this question as there are points in RP.

Ex. 5. What quantities of wines at 8*s.*, 10*s.*, 15*s.*, and 16*s.* a gallon respectively, must a vintner mix so as to obtain a mixture of 39 gallons worth 13*s.* a gallon?

Sol.—The ratio in which the wines are to be mixed is 3 : 2 : 3 : 5, which is the ratio obtained in *Example* 4.

Now 3 + 2 + 3 + 5 = 13. Since the total quantity in the mixture is 39 gallons, which is 3 times 13 gallons, he must take 3 × 3 or 9 gallons of the first and third kinds, 3 × 2 or 6 gallons of the second kind and 3 × 5 or 15 gallons of the fourth kind.

Ex. 6. The wholesale price of knives is 10 per cent. lower than the retail price, and 22 knives are counted into the score. What percentage is obtained by the retailer?

Sol.—The wholesale price is $\frac{9}{10}$ of the retail price, since the wholesale price is 90 when the retail price is represented by 100.

And the retailer gets 22 knives for the price of 20.

∴ The wholesale price of 22 knives = $\frac{9}{10}$ of the retail price of 20 knives = the retail price of $\frac{9}{10} \times 20 = 18$ knives

Thus, by retailing 22 knives, each at Re. 1 (suppose), the retailer would get Rs. 22, whereas he paid for them 18 knives' worth, *i.e.*, Rs. 18.

Thus he gains Rs. 4 on an investment of Rs. 18.

Hence his profit per cent. = $\frac{4}{18} \times 100 = 22\frac{2}{9}$ per cent.

Ex. 7. A person buys two kinds of paddy at Re. 1 9 as. and Re. 1 4 as. per kalam and mixes them; if he sells the mixture at Re. 1 7 as. per kalam making thereby a profit of $4\frac{6}{11}$ per cent., find the ratio in which the two kinds are mixed.

Sol.—The cost price of 1 kalam of the mixture = $(100 \div 104\frac{6}{11})$ of the selling price = $\frac{22}{23}$ of the selling price. And as the selling price of 1 kalam of the mixture is Re. 1 7 as., *i.e.*, 23 as. the cost price is 22 as.

∴ He must so mix the two kinds of paddy that the cost price of the mixture may be 22 as. per kalam.

For every kalam of the first sort taken, there is an increase of 3 as. over the average cost price.

But for every kalam of the second sort taken, there is a decrease of 2 as. below the average price.

∴ To compensate an increase of 3 as. in taking one kalam of the first sort, he must take $1\frac{1}{2}$ kalams of the second sort.

∴ He must take 1 kalam of the first sort with $1\frac{1}{2}$ kalams of the second sort.

Hence the ratio of the mixture is $1 : \frac{3}{2}$ or $2 : 3$.

Ex. 8 (*a*) A person bought 40 lbs. of tea and sold them for Rs. 80 gaining thereby the selling price of 5 lbs. Find the cost price of the tea per lb.

Sol.—The selling price of 40 lbs. = Rs. 80; and the gain is the selling price of 5 lbs. or Rs. 10.

∴ The cost price of 40 lbs. = Rs. 70; which comes to Re. 1 12 as. per lb.

Ex. 8 (*b*) I bought some tea at 2*s.* 6*d.* a lb. At what price should it be sold so that the selling price of 24 lbs. might be my gain on £15?

Sol.—Gain on £15 means the gain by selling tea worth £15.

But the quantity worth £15 $= \frac{15 \times 20 \times 2}{5}$ lbs. = 120 lbs.

The question means "how should I sell the tea so that by selling 120 lbs. I may gain the selling price of 24 lbs.?"

So, by selling (120 — 24 =) 96 lbs., the cost of 120 lbs. or £15 is realised.

∴ The selling price of 1 lb. = £15 ÷ 96 = 3*s.* $1\frac{1}{2}$*d.*

Ex. 9. A merchant has 60 lbs. of tea, a part of which he sells at 8 per cent. profit and the rest at 20 per cent. profit. He gains 12 per cent. on the whole. Find how much is sold at 20 per cent. profit?

Sol.—If he had sold the first part at 12 per cent. profit he would have got 12 — 8 or 4 pet cent. more profit.

And for the same quantity of the second sort, his profit would have been (20 — 12 or) 8 per cent. less.

But the lower profit in the 2nd instance is made up by the higher in the 1st.

So, Rs. 200 worth of the 1st kind of tea must be mixed with Rs. 100 worth of the 2nd kind, as only then the Rs. 8 additional profit in the 1st case will be equal to and make up for the Rs. 8 decrease in the profit in the 2nd case.

$\therefore$ the first : the second :: 200 : 100 or 2 : 1.

$\therefore$ the quantity sold at 20 per cent. profit $= \frac{60}{3}$ or 20 lbs.

Ex. 10. A tradesman mixes two teas in the ratio of 4 : 5, and by selling the mixture at $2\frac{7}{9}s.$ per lb. finds that he neither gains nor loses. If the second sort of tea is worth $\frac{3}{5}d.$ per lb. more than the first sort, find the price of each sort.

Sol.—The cost of 5 lbs. of the second sort = that of 5 lbs. of the first sort + $5 \times \frac{3}{5}d.$ Thus the price of 4 lbs. of the first sort and 5 lbs. of the second sort = that of 9 lbs. of the first sort + $3d.$ = the cost of 9 lbs. at $2\frac{7}{9}s.$, that is, $25s.$

$\therefore$ the price of 9 lbs. of the first sort $= 25s. - 3d. = 24\frac{3}{4}s.$

Hence the first sort costs $24\frac{3}{4}s. \div 9$ or $2s.\ 9d.$ per lb.

N.B.—The proofs of the following statements are left as exercises to the student.

(1) If the price of an article be changed in the ratio of $a : b$, then quantity obtained for a given sum of money will be altered in the ratio of $b : a$; and *vice versa.*

Thus if the cost price of 20 articles = the selling price of 18 articles, then cost price : selling price = 18 : 20 or the rate of profit = $11\frac{1}{9}$ per cent.

(2) If two kinds of an article are worth Rs. a and Rs. b each respectively, and if the mixture costs Rs. c, then

(i) the ratio of the mixture $= c - a : b - c$.

(ii) c lies between a and b.

(iii) c is nearer to the price of that sort, of which a larger quantity enters into the mixture.

Exercise XVI (c).

1. A merchant sold 40 lbs. of coffee at a gain of 12 per cent. and 80 lbs. more at a profit of 7 per cent. If he had sold the whole at a uniform profit of 10 per cent. he would have received Rs. 2 more. What did he pay for the coffee per lb.?

2. A merchant buys goods and sells one half at a again of 10%, one third at a gain of $16\frac{2}{3}\%$ and the remainder at a loss of $13\frac{1}{3}\%$; if the entire gain be £44 4*s.* 4$\frac{1}{2}$*d.*, what is the cost price of the goods?

3. A grocer had 150 lbs of tea of which he sold 30 lbs at 3*s.* 7*d.* per lb. and found that he was gaining only $7\frac{1}{2}\%$; by how much per cent. must he raise the selling price so that by selling the rest at the increased rate, he may gain 10% on his outlay?

4. A stationer bought 8 reams of paper for Rs. 20-8-0, the carriage came to Rs. 1-2-6. What will be his gain if he sells the outside quires of each ream at $2\frac{1}{4}$ annas a quire and the rest at $3\frac{1}{2}$ as. a quire?

5. A publisher sells his books to a bookseller at 10% profit; the book-seller sells the books to a customer at 25% profit. How much per cent. more than the cost price does the purchaser of the books pay?

6. A reduction of 30% in the price of apples would enable a purchaser to obtain 120 more for a sovereign. What was the price before the reduction?

7. A merchant bought a certain quantity of coal and sold one half of it at a price which was 10 per cent. above the cost price, the remainder being sold at 9*s*. 2*d*. per ton. If on the whole transaction he gained $12\frac{1}{2}$ per cent., find what was the cost price per ton.

8. Equal quantities of two teas whose values are as 3 : 4 are mixed together; if by selling 100 lbs. of the mixture for £20, a person gains $14\frac{2}{7}$ per cent., find the price of each sort.

9. A merchant buys whisky at 17*s*. 6*d*. a gallon and after adding water to it, sells the mixture at 18*s*. a gallon, gaining thereby 20 per cent. on his outlay. How much water did he add to each gallon of the whisky?

10. A tradesman buys 3 cwt. of sugar for £11, intending to gain $\frac{1}{11}$ of his outlay by selling it; but a guinea's worth at this calculation is damaged. At what price per cent. must he sell the remainder to gain 16·5 per cent. on his outlay?

11. A contractor bought 1600 parahs of unslaked lime at Ayyampet at 1 a. 6 p. per kalam; on slaking it every parah gave $3\frac{1}{2}$ cub. ft. of lime but of this $\frac{1}{14}$ was unserviceable. The carriage of the remainder to Kumbakonam (distant 15 miles) where it was required, cost 1 pie per cubic foot per mile. At what rate per cubic foot must he sell it at Kumbakonam to gain $21\frac{2}{3}\%$ on his outlay?

12 A smuggler had a quantity of tea which he expected would produce £855; after he had sold 20 lbs a revenue officer seized $\frac{1}{3}$ of the remainder, in consequence of which the smuggler made but £615. Required the quantity he had and the price per lb.

13. The wholesale price of knives is 23 per cent. lower than the retail price and 21 knives are counted into the score. What percentage is obtained by the retailer?

14. The real cost of an article is 56 per cent. of the price at which it is marked for sale. A reduction of $12\frac{1}{2}\%$ is however allowed for cash payment. How much does the seller gain per cent. by adopting the cash system?

15. How much per cent. must a tradesman add on to the cost price of his goods that he may make 10 per cent. profit after allowing his customers a reduction of 4 per cent. on his bill?

16. A tea merchant allows 10% discount off the marked price. On tea, marked Re. 1 4 as. per lb., he makes a profit of $12\frac{1}{2}\%$. The duty on tea is reduced 1 anna per lb., and the dealer then marks his tea at Re. 1 3 as. per lb. Find his percentage of profit now. (Discount being allowed at the same rate as before).

17. How much coffee at Rs. 3, Rs. 2 and Rs. $1\frac{1}{2}$ per viss respectively with 20 viss at Rs. 5 per viss will make a mixture worth Rs. 2-10-8 per viss?

18. A goldsmith has four sorts of gold which are 24, 10, 18 and 15 carats fine respectively and wishes to have 126 oz. of 17 carats. How much should he take of each sort?

19. A person melts 12 oz. of gold 21 carats fine and 18 oz. of gold 16 carats fine, 10 oz. of gold 17 carats fine and 5 oz. of gold of unknown fineness. If the mixture be 18 carats fine, find the degree of fineness of the 5 oz. of gold.

20. A vessel is filled with a liquid 3 parts of which are water and 5 parts syrup. How much of the mixture must be drawn off and replaced with water so that the mixture may be $\frac{1}{2}$ water and $\frac{1}{2}$ syrup?

PARTNERSHIP.

§ **120. A Firm** is an association of persons for the purpose of carrying on some business. They generally agree to share the profits and losses according to the *capital*

or sum of money invested by each, and the time it is permitted to be employed. Such persons are said to be in *Partnership*.

There are two kinds of partnership:—*Simple Partnership* and *Compound Partnership*.

Simple Partnership supposes all the investments to be made for the same time, and the gain or loss to be divided among the partners in proportion to the capital furnished by each of them.

Compound Partnership supposes the several sums of money advanced for the purposes of trade to be continued for unequal times, or to be increased or diminished at pleasure at any time during the continuance of such partnership. In this case the gain or loss resulting from the partnership is divided among the partners not only in proportion to the capital furnished by each, but also to the time for which their money has been employed.

EXAMPLE 1.—A began business with £1,000; 3 months after, he was joined by B with £500; 9 months after B joined, their profits amounted to £330. Find the share of B.

Sol. A put in £1,000 for 12 months. This is the same as putting £12,000 for one month. B put in £500 for 9 months—that is, £4,500 for one month. ∴ they put in amounts so that the profit should be divided in the ratio of 12,000 : 4,500, that is, in the ratio of 8 : 3.

B had therefore $\frac{3}{11}$ of £330 = £90.

EXAMPLE 2.—A and B hire a pasture for Rs. 23. A puts in 15 cows for 3 months and B 12 cows for 2 months. How should they divide the rent?

Sol.—A's putting in 15 cows for three months may be considered as equal to his putting in 15 × 3 cows for one month.

Similarly B's putting in 12 cows for 2 months may be taken to be equal to his putting in 24 cows for 1 month.

Hence the rent is to be divided in the ratio of 45 : 24; that is, in the ratio of 15 : 8.

Thus A has to pay Rs. 15, and B Rs. 8.

EXAMPLE 3.—A and B engage in trade, their capitals being as 15 : 7; at the end of 4 months A withdraws $\frac{1}{5}$ of his capital; B withdraws $\frac{1}{7}$ of his capital at the end of 5 months, but at the end of 8 months adds 5 times what A had withdrawn. How should a profit of Rs. 1,465 be divided between A and B at the end of the year?

Sol.—If we represent A's capital by 15, B's will be represented by 7. At the end of 4 months A withdraws ($\frac{1}{5}$ of 15 or) 3.

B withdraws ($\frac{1}{7}$ of 7 or) 1 at the end of 5 months, but adds at the end of 8 months (5 × 3 or) 15.

Thus A has put in 15 for 4 months and 12 for 8 months—that is, 156 for 1 month.

And B has put in 7 for 5 months, 6 for 3 months and 21 for 4 months—that is, 137 for one month.

Thus the profit of Rs. 1,465 is to be divided in the ratio of 156 : 137.

Hence A gets (Rs. 1,465 × $\frac{156}{293}$ or) Rs. 780, and B gets (Rs. 1,465 × $\frac{137}{293}$ or) Rs. 685.

Exercise XVI (d).

1. Three cowherds, A, B and C hired a pasture for £25 8*s.* 6*d.* A put 25 cows in it for 5 weeks, B, 12 cows for 3 weeks, and C, 7 cows for 7 weeks. What part of the rent had A to pay?

2. Two persons A and B hire a pasture for Rs. 94. A puts in 20 oxen for 5 months and B, a certain number of sheep for 8 months. Supposing that 2 oxen eat as much as 7 sheep, and that B pays a rent of Rs. 24, find the number of sheep put in by him.

3. A began to trade with a capital of Rs. 100. After 6 months he was joined by B with a capital of Rs. 150. Their profits at the end of the year amounted to Rs. 70. What were their respective profits?

4. A began business with £1,000; 2 months after, he was joined by B with £400; 6 months after B joined, their profits amounted to £910. How much more of the same profit would B have received if he had joined at the commencement?

5. A began trade with a capital of Rs. 840; 2 years after, B joined him with a certain sum of money. 6 years after B joined the business, their profit was $2\frac{1}{2}$ times as great as it would have been if A had managed alone. Find the sum invested by B.

6 A and B rent a field for £130. A puts in 8 horses for $2\frac{1}{2}$ months, 20 oxen for $1\frac{1}{2}$ months, and 80 sheep for 2 months; B puts in 50 horses for one month, 8 oxen for $2\frac{1}{2}$ months, and 60 sheep for 3 months. If the food consumed in the same time by a horse, an ox, and a sheep be as 3 : 2 : 1, find the portion of the rent which A and B must pay.

7. A, B and C rented a field of grass for £17 3*s.* for 20 weeks, and put into it 4, 5 and 6 score sheep respectively. After 2 weeks, A added 2 score, but at the end of 9 weeks removed all; B withdrew 1 score after 3 weeks and added 6 score at th end of the 5th week. What had each to pay?

8. Two persons, A and B, invest £4,000 and £6,000 respectively in trade. A is to have 20 per cent. of the profits for

managing the business, and the remaining profits are to be divided in the ratio of the capitals contributed by each; the entire profits at the end of the year amount to £1,000. How much does A receive more than B?

9. Three persons, A, B and C enter into partnership; A contributes £840, B £1,200, C £1,560. A acts as manager, for which he receives one-tenth of the profits, and the remainder of the profits is to be divided among all the three partners in proportion to their contributions. What fractional part of the whole profits does each receive?

10. A is a working, and B a passive, partner in a book-selling business. Their capital amounts to £6,400; of which £2,400 belongs to A, the rest to B. The profits are to be divided in the ratio of their capitals, but B should give A 5 per cent. of the profits for managing the business. If A's profits be £240 less than B's, find the total profit.

11. A, B and C enter into partnership, A contributing £5,000, B £3,000, and C £2,500, of the capital on the understanding that, after allowing one-eighth of the profits to C as manager, the remainder should be divided amongst them all in proportion to the amount of capital contributed by each. At the end of the year C receives £280. What are the total profits of the business? and how much do A and B receive respectively?

12. A and B enter into partnership on the following terms:—(1) B is to put in half as much capital again as A; (2) A, who is the working partner, is to receive out of the profits £8 for each £100 of his capital and B is to receive £4 for each £100 of his capital; (3) the rest of the profits are to be divided between them in the ratio of their respective capitals. At the end of the first year, A receives in all £2,456 and B £3,084. What capital did each put into the business?

13. A and B enter into a partnership with £700 and £900 respectively. At the end of a year they each took out £200, and C joined the firm with a capital of £500. At the end of the third year they dissolved partnership, and the net profit was found to be £350. What should each receive?

14. A has a capital of £666 13*s*. 4*d*. in a business for 12 months. B has £500 in the same business for 7 months, which he raises to £766 13*s*. 4*d*. for the remaining 5 months. C has £800 in the same business for the first 3 months, and only £300 for the last 9 months. If B's share of the profits is £53 0*s*. 10*d*. more than C's, what should be A's share ?

15. A and B enter into a speculation ; A puts in £60 and B £50 ; at the end of 4 months, A withdraws $\frac{1}{3}$ of his capital, and at the end of 6 months, B withdraws half his capital. At the end of 5 months, C joins them with a capital of £70. At the end of 12 months their profits amount to £120. How ought this to be divided among them ?

16. A and B engage in trade, their capitals being as 5 : 7 ; at the end of 4 months A withdraws $\frac{1}{7}$ of his capital ; B withdraws $\frac{1}{2}$ of his capital at the end of 3 months, but at the end of 7·5 months adds 6 times what A had withdrawn. How should a profit of £1,765 be divided at the end of the year ?

INSURANCE.

§ **121.** **Insurance** is the act of providing against a possible loss by entering into a contract with a company which undertakes for a certain percentage of the value to secure property from fire, risks at sea, &c.

Premium is the sum of money paid for insurance, and is reckoned at so much per cent. on the value of the goods insured.

" The person who undertakes to pay in case of loss is termed the *insurer* ; the danger against which he undertakes, the *risk* ; the person or property protected, the *insured* ; the sum which he pays for the protection, the *premium* ; and the contract itself, when reduced to form, the *policy*. "

There are many varieties of insurance, viz., *Life Insurance*, of which there are an innumerable variety of forms and premiums which are generally fixed by calculations based upon certain assumptions as to rate of mortality, rate of interest, &c. *Fire Insurance*, *Marine Insurance*, &c., &c.

EXAMPLE 1.—A house which costs Rs. 8,000 lets for Rs. 470 per annum. The outgoings for insurance amount to $\frac{3}{4}$ per cent. on the cost of the house. What rate of interest does it pay?

Sol.—The premium of insurance = Rs. 8,000 × $\frac{3}{4}$ ÷ 100 = Rs. 60.

The interest obtained = the rent for the house diminished by the premium of insurance = Rs. 470—Rs. 60 = Rs. 410.

And as the house costs Rs. 8,000, the rate of interest = (Rs. 410 ÷ 8,000) × 100 = $5\frac{1}{8}$ per cent.

EXAMPLE 2.—If a person, who pays £36 for insurance at $1\frac{1}{2}$ per cent., clears, in case of loss, the premium of insurance as well as the value of the goods he insures, find the actual value of the goods.

Sol.—If he pays £1 10*s.* for insurance, the insured value of the goods will be £100.

But the person insures the goods for such a value that, in case of loss, he may realise the premium of insurance and the actual value of the goods; thus while £100 is the insured value of the goods the actual value is only £100 diminished by the premium of insurance, that is, £100 — £$1\frac{1}{2}$ or £$98\frac{1}{2}$.

Hence if £$1\frac{1}{2}$ be the premium of insurance, the actual value of the goods is £$98\frac{1}{2}$.

And as £36 is the premium paid, the actual value of the goods = £$98\frac{1}{2}$ × 36 ÷ $1\frac{1}{2}$ = £2,364.

Exercise XVI (e).

1. What premium must be paid for insuring goods to the amount of £640 at 5 %?

2. A man who has a house worth £800 insures it at four-fifths of its value at 6*s*. 8*d*. for £100. What does his yearly premium amount to?

3. A house which costs £1,200 lets for £63 a year. The out goings for insurance amount to $1\frac{1}{4}$ °/₀ on its cost. What rate of interest does it pay?

4. A merchant paid £63 at 5 °/₀ for the insurance of goods which were lost. He cleared the value of the goods and the premium of insurance. What was the value of the goods?

5. If goods worth £1,312 be insured at £1 12*s*. per £100, to what amount must they be insured, so that in case of loss, the party insuring may recover the value of the goods and the premium?

6. A person paid £45 at the rate of $4\frac{1}{2}$ °/₀ for the insurance of some goods which were lost. He cleared the value of the goods and the premium of insurance. What was the value of his goods?

7. A ship and its cargo are insured for £6,000 which covers the expenses; the premium of insurance is £5 3*s*. 6*d*. per £100, and the duty, 6*s*. 6*d*. per £100. Find the value of the ship with its cargo.

8. A man insures property at 3 °/₀ for an amount sufficient, in case of loss, to repay the value of the property and the premium. If the latter amounts to £492, what is the value of the property, and for how much does he insure it?

9. A man insures his life for £700 at the rate of £1 19*s*. 7*d*. per £100. After a few years he insures for an additional £500, but at a higher rate. If the premium on the £700 and on the £500 be the same, what is the rate in the latter case?

10. A cargo is valued at £5,270 6*s*., the premium of insurance is at the rate of 5 guineas per £100, policy duty at 4*s*. per £100, and commission $\frac{7}{16}$ °/₀; for what sum must the cargo be insured to clear the value of the cargo and the expenses of insurance, and what premium must be paid?

EQUATION OF PAYMENTS.

§ **122.** When a person owes another several sums of money due at different times the process by which we determine the exact time when all these sums may be paid together without loss to either party is called the **equation of payment.**

Ex. 1. Find the equated time in the following:—Rs. 400 due 4 months hence, Rs. 600 due 8 months hence, and Rs. 700 due 6 months hence. We assume that the sum of the interests of the several debts for their respective times = the interest of the sum of the debts for the equated time.

Let x months be the equated time of payment.

Int. on Rs. 400 for 4 months + int. on Rs. 600 for 8 months + int. on Rs. 700 for 6 months = int. on Rs. (400 + 600 + 700) for x months.

i.e., int. on Rs. 1600 for 1 m. + int. on Rs. 4800 for 1 m. + int. on Rs. 4200 for 1 m.

= int. on Rs. $1700 \times x$ for 1 m.

$\therefore 1600 + 4800 + 4200 = 1700 \times x$

$\therefore x = \frac{106}{17} = 6\frac{4}{17}$.

$\therefore$ the whole is due at the end of 6 m. 7 days.

And, in general, if D_1, D_2, D_3.....be the amounts respectively due at the end of T_1, T_2, T_3.....months, then the equated time of payment $= \frac{D_1 \times T_1 + D_2 \times T_2 + D_3 \times T_3 +}{D_1 + D_2 + D_3 +}$ months.

Ex. 2. Find the equated time in the following case :—

(1) Goods worth Rs. 400 bought on May 1—Term of credit 3 months.

(2) Goods—Rs. 500 bought on May 15—Term of credit 4 months.

Goods worth Rs. 700 bought on June 10 at 3 months credit.

The sums are due as follows :—August 1, September 15, September 10.

Let August 1 be the starting date, (for convenience sake).

$\therefore$ Rs. 400 due in 0 days

$\therefore$ Rs. 500 ,, 45 days—from August 1 to Sep. 15.

$\therefore$ Rs. 700 ,, 40 days— do Sep. 10.

$\therefore$ The equated term $= \frac{400\times 0+500\times 45+700\times 40}{400+500+700}$ days

$= \frac{22500+28000}{1600} = \frac{225+280}{16} = \frac{505}{16} = 31\frac{9}{16}$ days

the equated time of payment is 32 days from August 1, *i.e.*, 2nd September.

Ex. 3. Find the equated time in the following example:—

Dr.

April 1, Rs. 400, due 2 months hence.

July 10, Rs. 600, due 1 month hence.

Cr.

May 1, by cash Rs. 250.

June 1, by cash Rs. 350.

The same rate of interest may be assumed throughout.

The total interest in either case should be the same.

Take April 1st as the starting date.

Interest on Rs. 400 for 61 days + interest on Rs. 600 for 131 days. = Interest on Rs. 250 for 30 days + interest on Rs. 350 for 61 days + interest on the balance, *i.e.* (600 + 400 — 250 — 350) for x days.

where x days is the equated time of payment.

$\therefore \quad 400\times 61+600\times 131=250\times 30+350\times 61+400\times x$

$\therefore \quad \dfrac{400\times 61+600\times 131-250\times 30-350\times 61}{400}=x$

$$\frac{24400+78600-7500-21350}{400}=x$$

$$\frac{103000-28850}{400}=x$$

$186 = x$. $\therefore$ the equated time of payment is 186 days from April 1, *i.e.*, the 4th October.

Exercise XVI (f).

1. A owed B Rs. 1000 to be paid at the end of 8 months; he paid Rs. 400 in 5 months; when is the remainder due?

2. Find the equated time of payment of Rs. 1200 due 4 months hence, Rs. 3000 due 6 months hence and Rs. 1800 due 8 months hence.

3. A person buys goods on March 10 to the amounts and on credit as stated :—

Value of goods Rs. 600, credit 3 mon.
..... 800 ..... 2 ,,
..... 900 ..... 4 ,,

Find the average number of days' credit each Rupee receives, if the goods are paid by a single payment of (Rs. 600+Rs. 800+Rs. 900).

4. A person buys goods as follows :—

Value of goods Rs. 950, credit 2 months
..... 864 ..... 1 month.
..... 375 ..... 70 days.
..... 1050 ..... 74 days

Date of purchase May 12. Find the average number of days' credit each Rupee receives, if the goods are paid for by a single payment of the nominal sum of the value of the goods.

REVISION PAPERS—III SERIES.

I.

($\pi = 3{\cdot}1416$)

1. Fig. 156 shows a semi-circular glass structure placed above a rectangular window, find the area of the red glass at 4 as. a square foot.

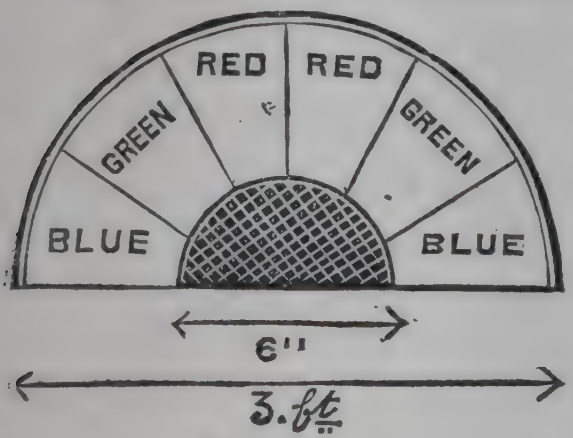


FIG. 156.

2. Calculate the banker's discount on a bill for Rs. 287 8 as. 6 ps. due on June 12th and discounted on April 15th at 4½ per cent.

3. A man has capital invested in 2½ per cent. stock at 98, 3½ per cent. at 85, and 4 per cent. stock at 102. Find what percentage of his capital must be invested in each stock in order that his income from each investment may be the same.

4. A and B are copyists in an office and copy the same manuscript. A writes 5 per cent. more lines than B in one hour, but works for 6 per cent. less time in a day. Will A turn out more or less work than B in a day and by how much per cent. ?

5. A stream flows from A to B, 8 miles distant at the rate of 2½ miles per hour. Two men, who each can row at the rate of 6 miles per hour in still water, start simultaneously from A and B to row towards each other. When they meet each at once turns and rows back to his starting point and then again turns and rows to meet the other. Find the distance between the two meeting points and the interval of time between the two meetings.

6. A terrace is supported on sixteen pillars each being 6 ft. 4 in. high. The terrace is 7 ft. above the floor. The pillars are to be

supported on stone conical frusta the diameters of the end circles being 1 ft. and 1½ ft. Find the cost of the stone pieces at Re. 1·4 as. per cube foot.

7. A lantern consists of (1) a lower portion in the form of a frustum of a pyramid on a regular hexagonal base, a side of the lower rim being 6 in. and a side of the upper rim being 9 in. and the slant height 12 in., (2) an upper portion in the form of a frustum of a pyramid on a regular hexagonal base, a side of the lower rim being 9 in. and a side of the upper rim 2 in. and the slant height 5 in. Each lantern face of each frustum is made up of glass. Find the superficial area of the glass required for the lantern.

8. Find the equated time of paying the balance of these accounts :—

Dr.

1910 March	13.	To Goods	... Rs. 135	credit 3 months.
April	18.	„	... „ 97	credit 4 months.
May	11.	„	... „ 192	credit 6 months.

Cr.

1910 May	10.	By Cash	... Rs. 50.	
July	3.	„	... „ 46.	
Aug.	10.	„	... „ 52.	

II.

1. Recent excursions have shown that the Lake Manasarowa is circular in form and 50 miles in circumference. Find the area of the surface and to what extent the volume is increased when the rainfall over it is 2 inches.

2. A and B enter into partnership with unequal sums of money, it being agreed that each is to receive 6 per cent. on his capital and that any profit remaining over after this, shall be divided equally. At the end of the year A receives Rs. 115-12-0, and B Rs. 118-4-0 ; and it is found that B thus gets Rs. 16-4-0 more than he would have got if all the profits had been shared in proportion to the capital invested by each. Find what each man's capital is.

3. I buy Rs. 8,250 stock and after paying income-tax at 4 pies in the rupee have an income of Rs. 489-9-4. What rate per cent. was the stock paying ?

4. Goods are bought at Rs. 30·8-0 with four months' credit and sold forthwith at Rs. 35-12-0 with such credit as would give 8 per cent. profit on the cost. How much credit was given, reckoning interest at 4%?

5. A person in Paris owes another in Vienna 6000 Austrian florins; instead of paying direct, he pays through a broker in London and thereby gains in English money £1 3*s.* $2\frac{25}{125}$*d.*; the exchange between London and Paris being 25 francs for £1 and between London and Vienna 11·6 florins for £1. Find the rate of exchange between Paris and Vienna.

6. A gold circular medal has a thickness of $\frac{1}{10}$ of an inch and a diameter of 1 inch. Assuming that the specific gravity of gold is 19, find its weight in gramms.

7. The section of a water course is given in Fig. 157. If the surface velocity of the water is 3 miles per hour and mean velocity = surface velocity × 0·6, find the discharge in gallons per minute when the height H of the water is (i) 3 ft. (ii) 4 ft.

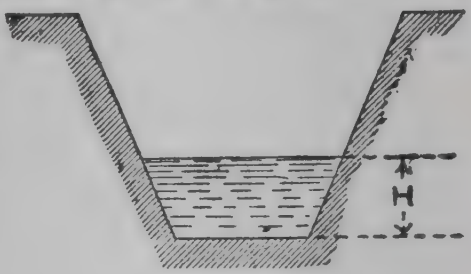


FIG. 157.

8. A train 88 yds.long moving at a uniform rate, overtakes a man who is walking along the line at the rate of 4 miles an hour and passes in 8 seconds. Twenty minutes after it has passed him, it meets a man walking in the opposite direction and passes him in 9 seconds. How long will it be after this before the two men meet?

III.

1. In order to find the bore of a piece of thermometer tubing mercury is let fall into it and it is found that, ·675 of a gram occupies 20 centimetres of the tube. What is the mean diameter of the bore (sp. gr. of mercury =13·6).

2. The ordinary shareholders of a railway company receive a dividend of 5 per cent. by the distribution of £70,000 after allowing 45 per cent. for working expenses, 15 per cent.

for the reserve fund and a debenture dividend of 6 per cent. on $\frac{1}{4}$ of the capital. Find the capital and receipts.

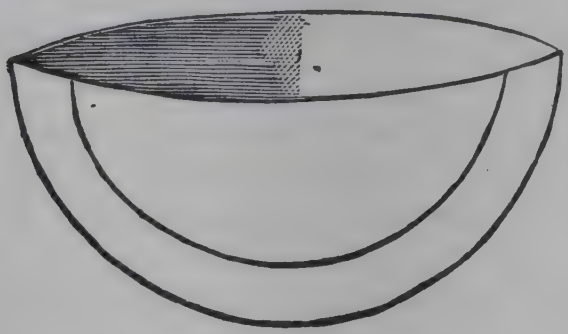


FIG. 158.

3. Find the quantity of water in a hemispherical bowl a section of which is given in Fig. 158 (scale $\frac{1}{25}$.)

4. Find, in French money, the price of a kilogram of an article which costs 1·5 shillings an ounce (25 francs = £1, gram = ·002204 of an ounce).

5. Water has to be diverted from a river through an eight-inch diameter pipe running full bore at a velocity of $2\frac{1}{2}$ feet per second to irrigate a field of 4 acres. How long will it take to deliver an inch of water over the whole area?

6. An alloy of silver is mixed with an alloy of gold in the ratio of 61 : 14; the percentage of lead in the silver alloy is 14·25 and that in the gold alloy is 15·75. What is the percentage of lead in the mixture?

7 The population of a town is 60,000. If the number of males were increased by 8% and the number of females by 5% the population would become 64,200. Find the number of males and females in the town.

8. A silver vessel of a cylindrical shape is ordered to be made. The inner diameter of the vessel is 6 inches. It holds a measure of water and the thickness of the silver is $\frac{1}{8}$ of an inch; assuming that one measure = 100 cub. in. and the s. g. of silver to be 10·5, express the weight of the vessel in tolas.

IV.

1. Find the area of a segment of a circle of which the chord is 15 ft. and height 2 ft. 6 inches.

2. The ordinates of an irregular piece of land are 2·5, 3·75, 4·5, 8·5, ·26, 12·35, 8, 10·25 and 2 yds. respectively the common interval is $1\frac{1}{2}$ yds. Find the area in square yds.

3. A person sells out $2\frac{3}{4}\%$ consols at $94\frac{1}{8}$ and by investing the proceeds in shares which pay an annual dividend of £4 per share raises his income by $4\frac{1}{2}\%$. Determine the price of each share to the nearest penny.

4. A bucket made of zinc sheet used for watering flower-pots is in the form of a frustum of a cone. The diameters of the ends are 9 in. and 15 inches and the axial height is 18 in. What quantity of water does it hold ?

5. The diagram represents a garden containing four semi-circular beds surrounded by paths as shown. It is drawn to a scale of 1 to 600. Find (i) the diameter of the beds in yards (ii) the area of the paths in square yards (iii) the cost of gravelling the paths at 12 as. per square yard.

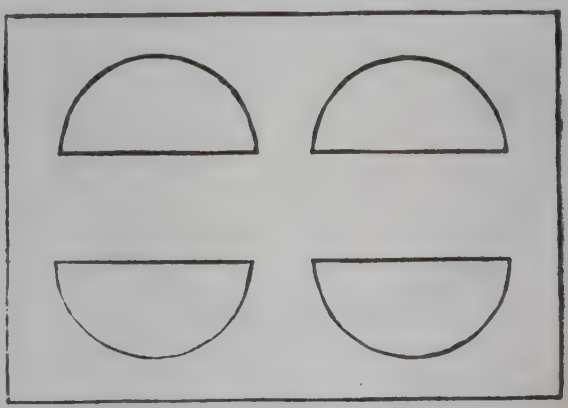


FIG. 159.

6. A, B, C start a business contributing to the capital in the proportion of 6, 9, 5 respectively. At the end of the first year A withdraws Rs. 1,500 of capital which is replaced by B. At the end of the second year B's share of the profit for the two years is Rs. 2,450 and C's Rs. 700. Assuming that the same profit was made in each year, find the amount of capital originally invested by each.

7. The railway between two stations A and B ascends at a gradient of 1 in 270 from A to a place C distant—$13\frac{1}{2}$ miles from A and then descends at a gradient of 1 in 160 from C to B. If the station at B is 30 ft. lower than that at A, find the distance from A to B. measured along the railway.

(A gradient of 1 in 60 means that the train rises vertically 1 ft. in travelling 60 ft.)

8. A piece of iron wire 15 metres long weighs 4·275 gr. in air and 35·75 grains in water. What is its diameter ? [Sp. gr. of iron = 7·2].

V.

1. If a certain piece of wire weighs 73·1245 grams and has a diameter of 0·400 cm., what is the diameter of a piece of the same length and material but which weighs 33·1285 grm.

2. The cross section of a graduated glass cylinder partly filled with water is given in the figure; scale ($\frac{1}{4}$). A ball is dropped into it and water rises by 2·4 cm. Find the volume of the ball.

FIG. 160.

3. A person bought the materials of an old house for Rs. 1,500 upon condition that they should be removed within 12 days under a penalty of Rs. $2\frac{1}{2}$ for every day of delay over the 12 days. He paid Rs. 43-12 as. a day for labour, and having sold the materials for Rs. 2681-4-0, he found that he had made a profit of Rs. 375. How many days did he take to remove the materials?

4. Find the commercial discount at $3\frac{1}{2}$ per cent. on Rs. 98714 as. 8 p. due 6 months hence.

5. Three stations A, B, C on a railway are passed by a train at 7, 7-50, 10-45; another train passes A and B at 8-50 and 9-15, both trains travelling uniformly, At what time should the second train pass C? If the first is travelling at 25 miles an hour, what is the rate of the second? (illustrate graphically).

6. A man invests Rs. 7,800 in $3\frac{1}{2}$ per cent. at 105? What amount of tax does he pay at 4 pies in the rupee and what income has he left?

7. Two adjacent sides of a rectilineal field contain an angle of 45°. 30′. A buffalo is tethered at the angular point. Find to the nearest foot the length of the rope so that the animal may graze on 2 acres of land.

8. A and B would take 30 days to do a piece of work. They work together for 10 days and then call in C and the three finish in

12 days more; C does in 2 days what A does in 3. Find their times separately for doing the work and thus compare them as workmen.

VI.

1. A cauldron used in a Hindu household is in the form of the accompanying figure (a cylinder standing on a spherical zone). Find the quantity of water it will hold to the nearest hundredths of a gallon, given that the diameter of the base is 8 in.

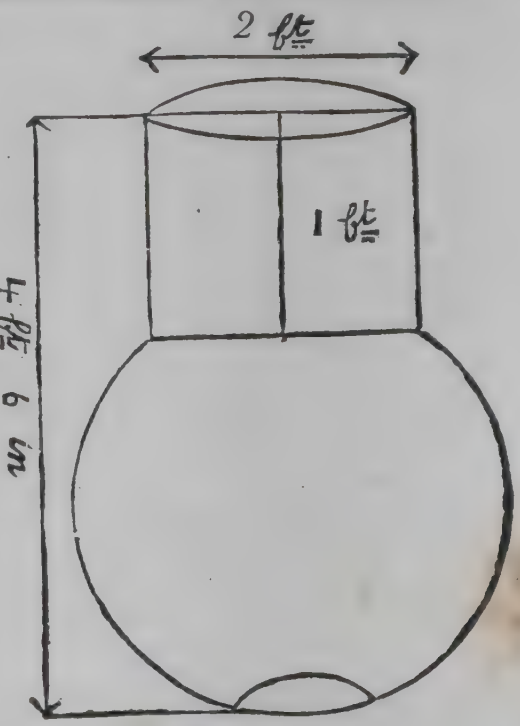


FIG. 161.

2. The radii of the circular faces of a frustum of a right cone are 26 in. and 16 inches and the area of its curved surface $420\pi\sqrt{5}$ sq. inches; find the height of the frustum.

3. Electric tram-cars start from a terminus every 5 minutes and when in motion run at 8 miles an hour. Stations are $\frac{1}{8}$ of a mile apart and there is a stoppage of $\frac{1}{3}$ minute at each. Find the greatest and least distance apart of two successive tram-cars (illustrate graphically).

4. A person changes Indian money into English at the rate of $1s.\ 4\frac{1}{8}d.$ per rupee and then changes the money back into Indian money at $1s.\ 4\frac{1}{16}d.$ per rupee. Does he gain or lose and how much does the difference amount to on Rs. 100?

5. How much stock at $96\frac{1}{2}$ must I sell in order to realise Rs. 965? If, by investing this Rs. 965 in $4\frac{1}{2}$ per cent. stock at 101, I increase my income by Rs. 8-12-0, what rate of interest was I receiving on the former stock?

6. A commences business with a capital of Rs. 3,000 and after 3 months takes in B as a partner with a capital of Rs. 500. Two months later, they take C into partnership with a capital of Rs. 400. At the end of the year, their net profits amount to 15 per cent.

on the capital invested. What should each receive of the profits ?

7. A retail book-seller is supplied by a wholesale firm which allows discount off the published price for certain books at $45^0/_0$; and all others at $35^0/_0$. Another wholesale firm offers a uniform discount of $37\frac{1}{2}^0/_0$. If $46^0/_0$ of the stock of books belongs to the former class and the rest to the latter, find which offer is more advantageous to him and by how much ?

8. A man who can row 4 miles per hour in still water, rows to a place 10 miles down stream and back again. How long does he take if the stream flow at the rate of 2 miles per hour ?

VII.

1. An observatory building consists of a cylindrical base of 15 ft. radius and 14 ft. height, surmounted by a hemisphere of the same radius. Find the cost of covering the outside with water-proofing material at Rs. 3-8-0 per square yard.

2. The figure below represents a vertical cross-wise section P Q of a stream of water flowing in a straight horizontal channel, the shaded portion representing water, the drawing being to the scale of 1 in. to 5 ft. 4 in. Find to two significant figures the number of gallons of water in half mile of the channel.

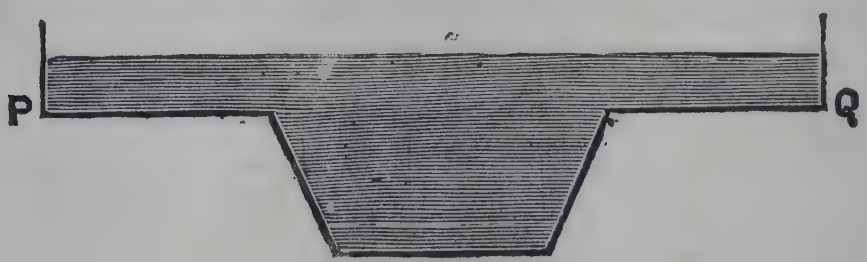


FIG. 162.

3. A and B run a mile race and A wins by 70 yds. ; afterwards B and C run a mile race and B wins by 70 yds. If A and C run a mile race, by how much would A win, it being assumed that each runs at the same rate as before ?

4. A man invests Rs. 2,080 in the $3\frac{1}{2}$ per cents at 98 and Rs. 3,500 in the 4 per cents, at $102\frac{1}{2}$, find the average rate of interest on his whole investment.

5. Three debts of Rs. 200, Rs. 300, Rs. 400 are due respectively in 3, 4, 5 months. If the interest is 5 per cent. per annum, what amount will pay them off at the end of 4 months ?

6. A refiner has 8 lbs. of gold 20 carats fine and melts it with 12 lbs. of gold 18 carats fine; how much pure gold must be put into it to make the whole 22 carats fine [pure gold is 24 carats fine, 20 carats fine means 20 parts gold and 4 parts alloy] ?

7. A railway company reduced passenger rates 8% on 1st class fares, 16% on 2nd, 24% on 3rd class. The increase in the number of passengers per mile was 18% in the 1st class, 24% in the 2nd, and 30% in the 3rd. How much did the company gain or lose per cent. on each class ?

8. A cyclist has to ride 56 miles. He rides for a time at 8 miles an hour and then alters his speed to 10 miles an hour and takes altogether 6 hours 12 minutes to cover the distance. Find when he changed his speed.

VIII.

1. A cask is in the accompanying form. Supposing it to consist

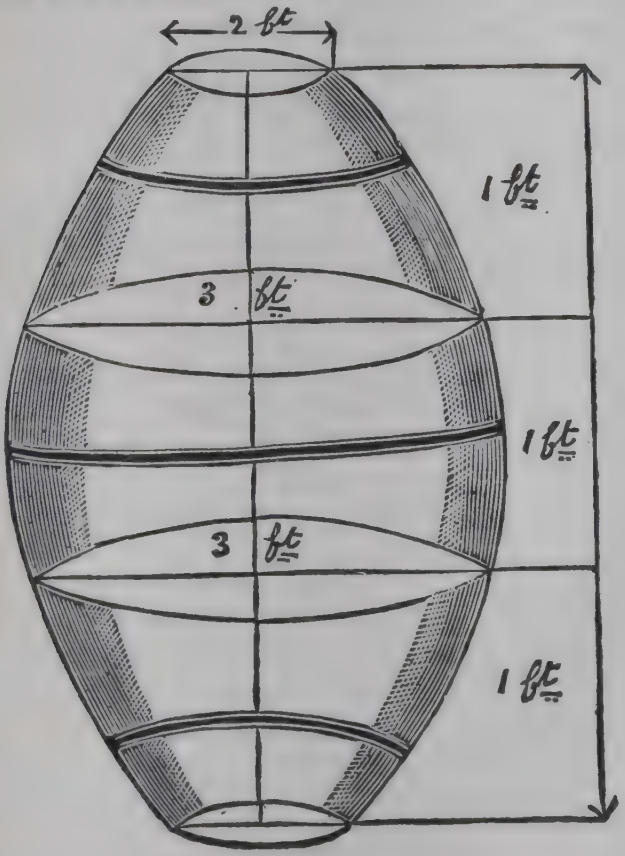


FIG. 163.

of (1) an upper conical frustum

(2) a spherical zone

(3) a lower conical frustum.

Find how many measures of rice a cub. ft. can contain if the cask holds 60 measures of rice.

2. The sides of the base of a triangular prism are 34, 50 and 56 ft. and its height is 40 ft. Find its volume.

3. Two trains start at the same time, one from A for B, the other from B for A. When they meet, the train from A has travelled 80 miles further than that from B; the former will reach its destination 4 hours, the latter 9 hours later. How far is it from A to B?

4. A fund-holder directs his broker to buy ten Rs. 500 Madras Bank shares quoted at Rs. 1,250 a share. To accomplish this, he authorises him to sell out Rs. 6,000 stock in the 3 per cents. at 95 and 7000 stcck in the $4\frac{1}{2}$ per cent. at 102. The broker's charge on each of the three transactions is $\frac{1}{8}$th per cent. What was due by the broker to the fund-holder?

5. English money being at a premium of $1\frac{1}{2}$ per cent., what English money to the nearest penny must a man remit to discharge a debt of 3000 francs, exchange being at the rate of 25·24 francs at par?

6. A cask is filled with spirit; $\frac{1}{2}$ of the contents is drawn off into another equal cask which is then filled up with water. The first cask is then filled up again out of the second one, and then the second one is filled up again out of the first. Find what proportion of spirit the second cask finally contains.

7. There is a well containing 1000 gallons of water; two pumps raising 25 and 35 gallons per minute respectively are employed to empty it; while it is constantly supplied by a spring which can refill it in 45^{m}. the two pumps work for 5 minutes, when that of the smaller capacity ceases to work for 10 minutes. The two pumps then work together until the well is empty. How long will each pump have been employed?

8. A bill falls due in 6 months. The creditor agrees to accept immediate payment of half the amount and to defer payment of

the other half for 1 year. He finds that by this arrangement, reckoning interest at 6 per cent. per annum, he gains Rs. 5-1-0. What is the amount of the bill?

IX.

1. The moon's diameter is 2200 miles and the earth's diamater 8000 miles. How many moons would it take to form a body of the size of the earth?

2. The surface of a sphere of radius r inches is $4\pi r^2$ square inches where $\pi = 3{\cdot}1416$. If π is taken, instead, to be equal to $\frac{22}{7}$, find roughly in square miles, the difference that this will make in the area of the earth's surface—the earth being a sphere of radius 4000 miles.

3. Equal sums are invested in 3½ per cent. stock at 90, 4 per cents. at 95, and 5 per cents. at 100. The price of each stock rises Rs. 5, then the first and third are sold and the proceeds invested in the second. Find the rise in income on the whole sum invested as a percentage. No brokerage is to be reckoned.

4. Two bills for Rs. 912-8-0 and Rs. 602-4-0 are due on June 5 and 30 respectively. Find their joint values on June 10, reckoning a banker's discount at 5 per cent.

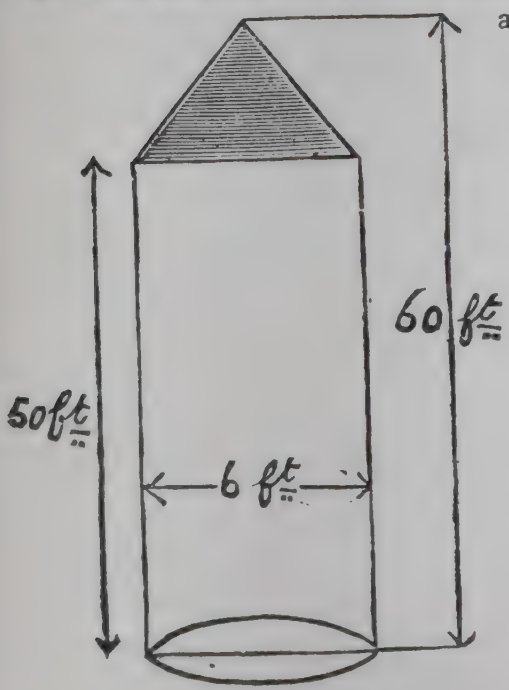


FIG. 164.

5. The legal weight of a 20 yen peice is 257·21 grains, the sterling value £2 0*s*.-4½*d*. The weight of an English sovereign being 123·27 grains. What is the fineness of the 20 yen piece?

6. Fig. 164 is that of a tower consisting of (*a*) a conical top portion standing on a cylindrical structure. The walls are 1 ft. thick. Find the cost of constructing it at Rs. 5-4-0 per cubic ft. allowing $\frac{1}{10}$ of the space for doors and windows.

7. A travels from P to Q a distance of 60 miles and back again at the rate of 9 miles an hour. On his way back he meets B who travels at the rate of 6 miles an hour and who started at the same time from P. Find graphically the distance of their meeting point from P.

8. If an article be insured by a single payment of 15 per cent. on the amount for which it is insured, to what amount should I insure an article worth Rs. 540 so that in the event of loss I may recover both the value of the article aud the insurance payment.

X.

1. Find the area of a segment of a circle of which the arc is one-third of the circumference, the radius being $7\frac{1}{2}$ inches ($\pi = 3\frac{7}{50}$, $\sqrt{3} = 1\frac{1}{19}$).

2. The height of a conical tent is $7\frac{1}{2}$ ft. and it is to enclose 200 sq. yds. of ground. Find how much canvas will be required ? $\pi = \frac{22}{7}$.

3. The present income of a railway company would justify a dividend of 8 per cent if all the shares ranked equally. But as £200,000 of the stock consists of preference shares which receive 9 per cent. dividend, the ordinary shareholders receive only 6 per cent. What is the whole amount of stock ?

4. A house is insured against fire at the rate of $\frac{1}{2}$ per cent. If the premium paid is Rs. 750, what is the amount for which the house is insured ?

5. A bill for Rs. 7,138-8-0 drawn on March 12th and payable 8 months after date was discounted by a banker on May 10th at $3\frac{1}{2}$ per cent. How much did the banker give for the bill ?

6. A chimney of the shape indicated in the figure is made of mica. The bulging portion being a zone of a sphere. Find the

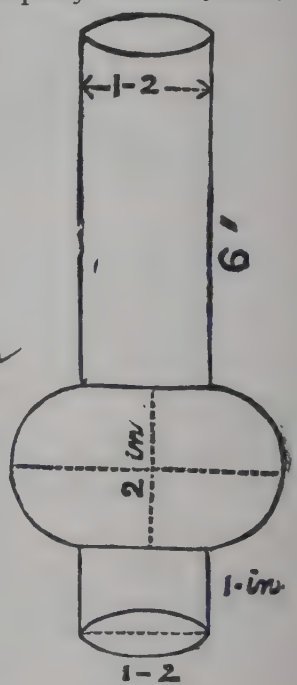


FIG. 165.

superficial area of the mica required for the purpose. [The diameter of the cylindrical portion is 1·2 in.].

7. Three persons contribute Rs. 300, Rs. 500, Rs. 750 respectively towards a venture on the understanding that profits shall be so divided that the rate of interest which each receives shall be proportional to the amount of his contribution. If the profits for the year amount to Rs. 250, how much will each receive?

8. The hands of a clock are put back $\frac{1}{4}$ hour between 7 and 8 o'clock thereby making the hands exactly coincide. What will be the true time when they are next together?

Miscellaneous Examples.

1. Sub-Inspectors in the Police Department are allowed horse allowance at the rate of Rs. 15 per mensem. For this amount they have to maintain a pony whose purchase value comes to Rs. 250. The cost of miaintaining the pony per month is detailed thus:

	Rs.	A.	P.
Gram	10	0	0
Rent for Stables	3	0	0
Grass	5	0	0
Syce	8	0	0
Shoeing	2	0	0
Casualties, medicine, etc. (on an average). ...	1	0	0

If the salary of a Sub-Inspector is Rs. 75 per mensem and if he has to pay interest at 9 % per annum, on the sum of Rs. 250 borrowed for the purchase of the pony, find what the net salary of his post is.

2. According to the Census figures, the number of widows under 10 years and above 5 is 4332 in this Presidency and under thirty there are 4 lakhs in round numbers. Of these among Hindus there are 673 out of the total 738 under 5 years and 4072 out of 4332 under ten years and under thirty about 3 lakhs and 75,000. Among Mussulmans there are 31 under 5 years and 161 under 10, and among Christians 19 widows under 5 years and 6 under 10. From these figures, prepare the following table:—

Widows.	Under 5.	Above 5 and under 10.	Under 30.
Hindus.			
Mahom-medans.			
Christians.			
Other nationalities.			
Total for the Presidency.			

Express each figure in the above table as a percentage of the coresponding total for the Presidency.

3. The following list of prices of white silk cloths is given in an advertisement :—

					RS.	A.
(1)	2 yards by	33 inches	... ...	Price	4	0
(2)	3 ,,	48 ,,	... ...	,,	9	0
(3)	3 ,,	54 ,,	... ...	,,	10	0
(4)	3½ ,,	48 ,,	... ...	,,	10	0
(5)	3½ ,,	54 ,,	... ...	,,	11	0
(6)	4 ,,	48 ,,	... ...	,,	11	0
(7)	4 ,,	54 ,,	... ...	,,	12	0
(8)	8 ,,	42 ,, Sari	... ...	,,	22	0
(9)	9 ,,	42 ,, Sari	... ...	,,	25	0

Prepare a list of prices for each yard of length. Hence find the cheapest and dearest sorts. Illustrate your answer graphically.

4. The total Jail population during the year 1911 was 18,277. The number of literate convicts admitted increased from 13·58 per cent. of the whole in 1910 to 15·83 per cent. in the year 1911. Assuming the total Jail population in 1910 was 13,042, find the decrease or increase per cent. in the number of illiterate convicts of the Jail.

5. The financial results during the year 1910 of the working of all the Railways in India are as follow :—The net earnings of all the Railways taken together amounted to Rs. 2644 lakhs as against Rs. 2398 lakhs in 1910 although there was an increase in working expenses of Rs. 168 lakhs. These net earnings work out a return on the capital outlay of 5·8 per cent. as against 5·46 in the previous year. Find the difference in gross earnings for the two years and also determine the capital outlay in as many ways as possible from the information given. How far do these results agree ?

6. The following is an extract from the S. I. Railway notifica-

tion published in a newspaper announcing a change in the time-table of trains ;—

No. 2 Ceylon Boat Mail.		Stations : Distance from Beach.			No. 1 Ceylon Boat Mail.	
Revised.	Present.				Present.	Revised.
8—0	7—55	A. Madras Beach 0		D.	17—45	17—40
7—45	7—40	D.	Madras, Egmore	A.	17—59	17—54
7—37	7—32	A.	3 miles	D.	18—28	18—23
6—54	6—47	D. / A.	Pallavaram	A.	...	...
			15 miles	D.	19—8	19—3
14—51	15—3	D.	Madura	A.	11—13	11—44
14—56	14—48	A.	348 miles	D.	11—30	11—59
9—58	10—0	D. Tuticorin 447 miles		A.	16—5	16—10

Read from bottom to top. Read from top to bottom.

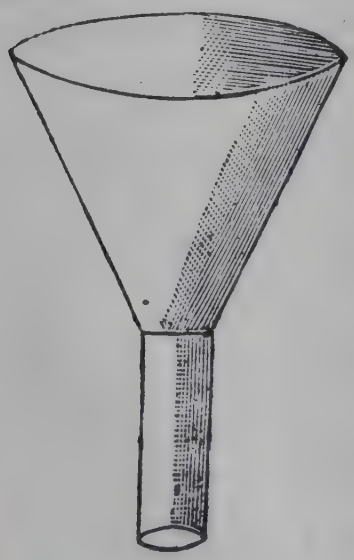


FIG. 166.

Draw graphs of

(1) No. 1 CeylonBoat Mail according to the present and revised time-tables.

(2) No. 2 Ceylon Boat Mail according to both the time-tables.

From a graphical consideration, find how the time and place of meeting of the two trains are affected by the change in the time-tables.

7. Find the cubical contents of the funnel shown in the adjoining figure (scale $\frac{1}{6}$.)

8. The following interesting figures are given in Mr. Noel Paton's Review

of the Trade of India in 1910-11 as to the imports of machinery into India :—

Machinery in 1910-11 represented only 17·6 per cent. of the total value of imported metals and manufactures thereof and only 3·7 per cent. of the imports of all merchandise. The total value of all classes of machinery has receded from Rs. 506·07 lakhs (£3,373800) to Rs. 473·04 lakhs (3,153,600), the decline being Rs. 33·03 lakhs (220,200), or 6·5 per cent. Examine the consistency of the figures and determine in Indian and English money (1) the total value of imported metals and manufactures (2) the value of imports of all merchandise in the year under report

9. Draw a statistical graph in the same figure from the following figures for (1) Cotton, (2) Jute :—

Year.	Cotton.	Jute.
	RS.	RS.
1907-8	10,76,66,000	18,29,76,000
1908-9	11,53,72,000	15,78,60,000
1909-10	11,91,56,000	17,09,65,000
1910-11	11,01,50,000	16,99,45,000
1911-12	9,77,89,000	16,60,66,000

Find when the rise was greatest in (1) cotton ; (2) jute.

10. The following is a table of clasification of people supported by agriculturists and actual workers in agriculture out of 1,000 for the years 1910 and 1911 :—

Heading.	Supported by agriculture.		Actual workers in agriculture.	
	1910	1911	1910	911
Non-cultivating land-owner.	34	30	23	19
Cultivating land-owner ...	461	512	426	484
Non-cultivating tenant ...	6	2	4	1
Cultivating tenant ...	225	167	207	151
Farm labourer ...	279	284	340	345

Find the decrease or increase per cent. in the figures for 1911 as compared with those for 1910 under each head.

11. "The figures for production of salt are not so encouraging as those for its consumption. The manufacture in the country has not been able to keep pace with the consumption. Whereas the *latter has increased by more than 10 million* maunds during the last 20 years, the former *shows an increase only of four*, consequently the importation has increased. The figures for production and importation are :

	Inported.	Manufactured.
1890-91	10,790,319	28,635,951
1909-10	13,781,826	32,336,407"

Examine the degree of the accuracy of the statement as an approximation.

12. In the year ending June 1911 the number of arrack shops in Mysore for the sale of country spirits was 781 or 43 less than in the previous year. The total consumption of arrack fell from 383,322 gallons to 375,756. By what percentage was the average supply to each arrack shop increased or diminished ?

13. Two sets of prices are given of Zerotherm flasks below ;—

	Rs.	A.	Rs.	A.
Pint size. Electroplated cover	7	4	5	15
Do. Solid Leather cover	9	8	7	8
Quart size. Imitation Crocodile cover ...	10	8	7	14
Do. Electroplated cover	13	8	11	
Do. Solid Leather cover	16	8	13	8
Pint Refills	3	12	2	15
Quart Refills	8	8	6	4

Show by means of a graph the relation between the corresponding prices of each flask.

14. The Governmet of Madras have approved the proposals submitted by the Director of Public Instruction for opening 108 elementary schools during the year 1912 in 41 Municipalities and have sanctioned the payment of a sum of Rs. 29,962 to the Municipalities concerned to meet the cost of the new schools. Find to the nearest pie the average sum intended to be expended.

on each elementary school to be opened and also the average amount of contribution to each Municipality.

15. The application of the new rates to all occupied ryotwari lands in the Taluq resulted in a net increase of assessment on wet lands of Rs 25,749 or 11·46 per cent and a decrease of assessment on dry lands of Rs. 851 or 1·22 per cent. or in a net increase of the aggregate revenue of Rs. 24,898 or 8·47 per cent. Find the original income on dry and wet lands as far as the data justify. (Each of the precentages is correct to two decimal places). In how many ways can the aggregate income be determined? How far do these results agree?

16. The quarterly premia that a National Insurance Company charges at different ages to secure Rs. 2000 payable after 15 years are given in the following table :—

Age next birthday.	25	30	35	40	45
Quarterly premium.	Rs. 32·6	Rs. 32·13	Rs. 33·7	Rs. 34·7	Rs. 35·13

(1) Draw a graph and find the premium at the age of 43.

(2) If a man insures his life at 25 and dies at the end of his 27th year, how much does his family gain? [Simple interest being calculated at the rate of 6 per cent. per annum.]

17. The initial pay of an educational officer might remain as at present Rs. 500 rising by annual increments of Rs 50 to Rs. 950 and then by annual increments of Rs. 100 to Rs. 1250. At this stage the pay of an officer might further increase from Rs. 1250 to Rs. 1850 by annual increments of Rs. 100. Supposing that the maximum of a grade and the minimum of the next higher grade, if same, should be drawn for two consecutive years, find when an officer entering Educational service at 25 on Rs. 500 will get Rs. 1850, earned by him during the period of his service.

18. The figure on the next page is a representation of a Zerotherm pint flask to the scale $\frac{1}{4}$. Assuming that the cylindrical portion only is filled with fluids, test if it is really a *pint* flask.

19. The following varieties of the Chatterton Patent Pum worked by one man are available :—

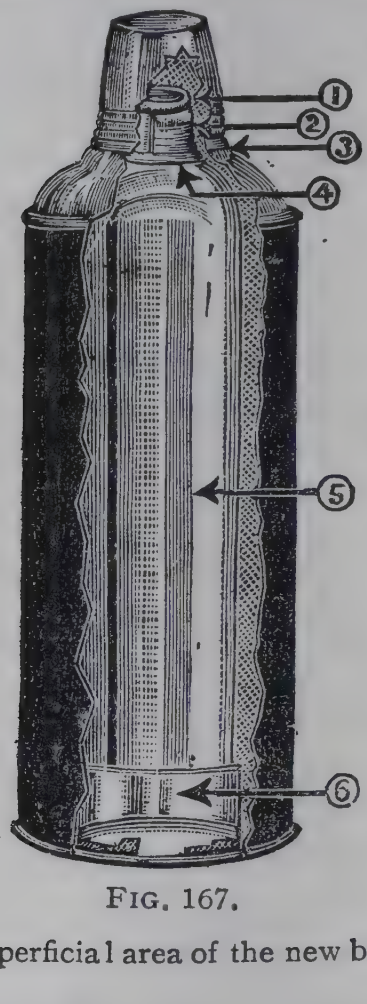


FIG. 167.

Max. height of lift.	Approx. discharge in gallons per hour.	Pric
5 ft.	2,400	Rs. 18
8 ,,	1,500	Rs. 14
12 ,,	900	Rs. 11
25 ,,	550	Rs. 10

Assuming that 1 gallon of wat weighs 10 lbs. and a horse power 550 ft. lbs. per second, calculate th horse power of each pump an examine if the prices are propo tional to the H. P.

20. The City Engineer of th Madras Corporation makes the fo lowing recommendation for increa ing its revenue :—"Increase th license fee for each new buildi application from Re. 1 per 400 sq. of floor to Re. 1 per 100 sq. Anticipated extra annual reven about Rs. 1,500." Calculate th superficial area of the new buildings under construction every yea

21. The average earning of a fly shuttle weaver is Rs. 12-8 month, while that of a country loom weaver is Rs. 10 a mont It is calculated that at a modest computation there cannot b any fewer than 10,000 fly shuttle looms in the Coast district Find the annual minimum increase in the income of the weavers

22. A merchant wishing to clear out his old stock so one lot of goods at a reduction of 12½ per cent., another at reduction of 25 per cent., and a third at a reduction of 50 p cent. on the usual prices. He realised Rs. 682-8, Rs. 1,012- Rs. 450 for these lots respectively, and found that on the who

he had a loss of $2\frac{1}{2}$ per cent. on the price paid by him for the goods. What would he have lost or gained per cent. if he had sold all the goods at a reduction of 20 per cent. on the usual prices?

23. On and from the 15th June 1912, the Passenger trains on the Bezwada-Masulipatam Section run to the following (distance 60 miles) timings :—

Station.	42 mixed mail.	48 mixed.	Station.	41mixed.	47mixed.
	H. M.	H. M.		H. M.	H. M.
Bezwada. Dep.	6-15	14-40	Masulip'tm. D.	9-5	15-35
Masulip'tm. Arr.	10-10	18-50	Bezwada. Arr.	13-40	19-40

Find graphically or otherwise (1) when and where trains No. 41 and No. 42 meet, (2) when and where No. 48 and No. 47 trains meet.

24. The positions of four points are specified thus : from A to B is 2·6 miles north and 3·2 east ; from B to C is 2·5 north and 1·3 west ; from C to D is $1\frac{1}{2}$ north 3·4 west and from D to E is 4·2 south and 1·2 east. Show on a plan, the points A, B. C, D, E on the scale of 1 inch to a mile.

25. The straight line PQ in the figure represents a level railway

FIG. 168.

carried through cuttings and on an embankment. The irregular line represents the section of the ground. The scale of horizontal distances is 1 inch to 10 miles, the scale of vertical distances 1 inch to 1,000 ft. Draw up a table showing the height and depth of the ground above or below the railway at intervals of 2 miles, using the signs + and −.

26. In order to determine the cross sectional area of a river

channel at a certain place, a series of soundings are taken and the results are as given below :—

Distance of sounding from left bank in feet.	0	8	12	14	18	20	30
Depth of sounding in ft.	3	12	15	19	17	12	0

Plot a section of the channel and calculate the area of the cross-section.

27. ABCD is a square field each side being 150 yds. AB runs south and AD east. A foot path runs across the field and one point of it is 25 yds. distant from BC and 40 yds. distant from AB. Another point of it is equidistant from AD and BC. Show by a diagram drawn on squared paper, to a scale of 5 yds. to a tenth of an inch, the exact position of the foot-path.

28. A pyramid stands on a horizontal square base of side 24 inches. The vertex is 9 inches vertically above the point of intersection of the diagonals of the base. Find the altitude of a face of the pyramid and also the length of a line joining the vertex to any angular point of the base.

29. A hay shed consists of a rectangular space 60 ft. long, 20 ft. high and 15 ft. wide with a semi-circular roof above it. Calculate how many tons of hay can be stored in it if a ton of hay occupies 290 cub. feet.

30. Find the value of:

$$1 + mn + \frac{m\,(m-1)}{1\cdot 2}\,n^2 + \frac{m\,(m-1)\,(m-2)}{1\cdot 2\cdot 3}\,n^3.$$

(1) When $m = 3$, $n = 1$ and (2) when $m = 8$ and $n = 3$.

31. A sphere S whose radius is 20 centimetres long is coated with a substance 1 mm. thick. Show that the volume of the substance is nearly equal to the surface of S multiplied by the thickness of the substance and find the percentage error involved in assuming this to be the case.

32. The co-ordinates of three points A, B, C are x_1, y_1; x_2, y_2; and x_3, y_3. Draw the ordinates and express the area of the Δ as

the sum or difference of trapeziums. Hence show that the area of the $\triangle$ is $\frac{1}{2}(x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3)$. Apply this result to find the area of a $\triangle$ whose co-ordinates are (3,5), (8,7), and (9,10).

33. A wooden tub of the size and shape indicated in Fig. 169 is filled with water for a bath. How many gallons of water are allowed for the bath? (Scale $\frac{1}{24}$).

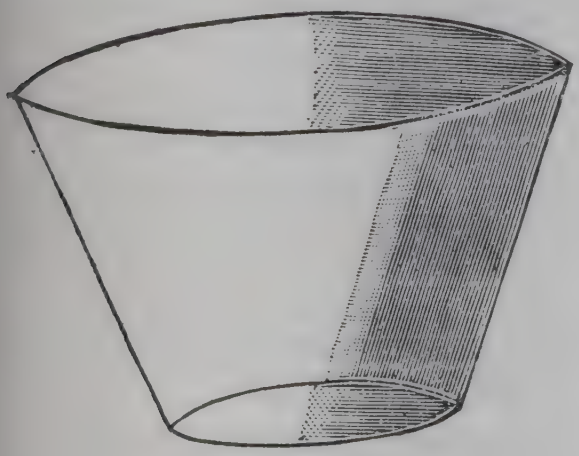


FIG. 169.

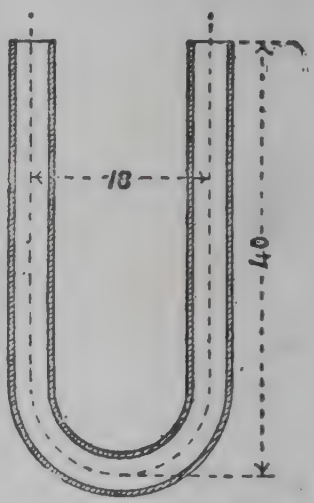


FIG. 170.

34. Fig. 170 is the section of a bent tube consisting of two straight lengths joined by a semicircle. the external diameter of the tube is 60 mm, thickness 3 mn. Find the volume of water it will hold when full. [The measurements given in the figure are centimetres].

35. The following is a time table of two trains running between Madras and Renugunta:—

Distance in miles.	Up train. H. M.		Down train. H. M.	
0 Madras	D. ...	10-0	Renigunta D. ...	9-20
26 Trivellore	A. ...	11-16	Arkonam A. ...	11-23
	D.	11-23	D.	11-55
43 Arkonam	A. ...	12-25	Trivellore A. ...	12-40
	D.	12-46	D.	12-46
84 Renigunta	A. ...	14-13	Madras A. ...	14-25

Plot in one diagram on squared paper graphs to show the positions of both trains at any time between those given, assuming that they run uniformly between the stations. When and where do the trains pass one another?

36. A gasometer is in the form of a cylinder with a rounded top and the following dimensions are given; diameter = 30 ft., height at edges = 15 ft., height in middle = 18 ft. Calculate the number of cubic feet of gas that the gasometer will hold. Assume the rounded portion to be a spherical cap.

37. L_1 and L_2 are two light houses 165 miles apart along a straight stretch of coast $L_1 L_2$. A ship S is off the coast and $L_1 S L_2$ is found to be 43^0. Also the nearest distance of the ship foom the coast is found to be 440 yards. Determine the position of the ship by drawing to scale (1 inch to one mile).

38. A barometer tube whose internal diameter is $\frac{1}{2}$ inch is 40 in. long and filled with mercury. Find the weight of the mercury, given that s. g. of mercury is 13. Assume $\pi = 3{\cdot}142$.

39. The area of a triangle is given by $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. Reduce this to its simplest form for a Δ in which $a^2 = b^2 + c^2$ and calculate the area when $b = 15{\cdot}4$ and $c = 18{\cdot}9$ inches.

40. The gas service pipe to a house 50 ft. from the main is $\frac{3}{4}$ inch in diameter. For how many burners each taking 4 cub ft. of gas per hour will this serve? The number of cubic feet per hour delivered by a pipe on that main is $\frac{1000}{3}\sqrt{\frac{20d^5}{\alpha}}$ where d is the diameter of the pipe in inches and α is the length of the pipe in yards.

41. On a map drawn on a scale of an inch to a mile two points are found to be 1·5 in. apart. One point is at the sea level; the other at a height of 1,800 ft. Find the actual distance between the points.

42. The time of swing of a pendulum is proportional to the square root of its length. If a pendulum 39 inches long swings once per second, how many swings per minute will be made by a pendulum 6 ft. long?

43. Extract $\sqrt{31}$ graphically in 3 different ways.

44. Multiply together $x+a$, $x+b$, $x+c$ and arrange your result by bracketing together terms containing like powers of x. Describe your result in words and deduce from it the result of multiplying $x-a$, $x-b$ and $x+c$ together.

45. A tunnel has to be bored through a mountain connecting two places P and Q. To get from P to Q over the mountain one had to walk 3 miles upwards at an angle of 25° and then 2 miles downwards at an angle of 35° with the horizon, the path being all in one vertical plane. Find the length of the tunnel, the difference of level of its two ends, and the inclination of the tunnel to the horizontal.

46. A chimney of solid brick in the form of a frustum of a cone is 240 ft. high; the radii of its base are 24 ft. external, 18 ft. internal, and of its top 12 and 10 ft. Find the volume in cub ft.

47. Compute to two significant figures the diameter and cross sectional area of a specimen of copper wire one mile of which weighs 3·9 lb. One cu. in. of copper weighs 0·35 lb.

48. It is desired to dig a trench 5 feet wide with vertical sides and horizontal base across a piece of ground the contour of which is given by the following measurements. Calculate in cubic feet the quantity of earth dug out.

Horizontal measurement along datum line in feet.	0	6	12	18	27	32	40	49
Height of surface above datum line in feet.	0	6	7	8	10	8	9	0

49. A man playing 5 holes of golf course walks first 240 yds. due east, then 120 yds., 25° south of east, then 280 yds. due south, then 220 yds. 45° west of north, then 200 yds., 25° west of south thus arriving at the fifth hole. Draw the plan of the course. on a scale of 100 yds. to the inch, and find from it how far the fifth hole is from the start.

50. P, Q, R are the mid. points of the sides of a $\triangle$. Construct the $\triangle$.

51. A cyclist can travel m miles an hour in a still day, but his

speed is decreased d miles an hour against a certain wind and increased i miles an hour with this wind. How long would it take him to go 15 miles and to return, going against the wind and returning with it?

52. The exposure in seconds required to take a photograph is proportional to the square of the stop number and inversely proportional to the speed number of the plate. It is found that $1\frac{3}{4}$ sec. is the correct exposure with an ordinary plate (speed 75) using a stop numbered 18. Find a formula connecting speed number, stop number and the exposure.

53. Two circles have radii 1·4 and 2·2 inches and their centres are 4·2 inches apart. Draw a circle of radius 1·2 in. touching them both.

54. The section of a tube railway is a segment of a circle greater than a semi-circle, the breadth of the tube at the level of the rails being 10 ft. and its greatest height above the rails, 15 ft. Draw the section on any convenient scale and write down the greatest breadth of the tube in feet to the nearest tenth.

55. On a steamer the expense of coaling per mile is found to be directly proportional to the square of the speed maintained. If it is Re. 1-12 as. per mile when the speed is 28 miles per hour, the other expenses being Rs. 27-8-0 per hour at all speeds, express in rupees the expense of a journey of x miles at s miles an hour.

56. A petroleum cylindrical tank has a diameter of 45 ft. and is 28 ft. high. The curved surface of the tank is made of rectangular iron sheets 4 ft. by $2\frac{1}{2}$ ft. Find the number of sheets required for the curved surface. The cylinder is surmounted by a spherical cap. The highest point of the rounded roof is 31 ft. from the base. Find the capacity of the tank in gallons (1 cub. ft. = 6·25 gallons).

57. The gauge of a railway in a customs house used for the transmission of goods on trucks is 4·5 ft. A change in the course of trucks at right angles is effected by the rotation of a circular disc (or turn table) as indicated in Fig. 171. The radius of the disc is 7·5 ft. What must be the maximum distance between the centres of the wheels of a truck moving on the railway? Find the area of the portion of the circle outside the track of the wheels.

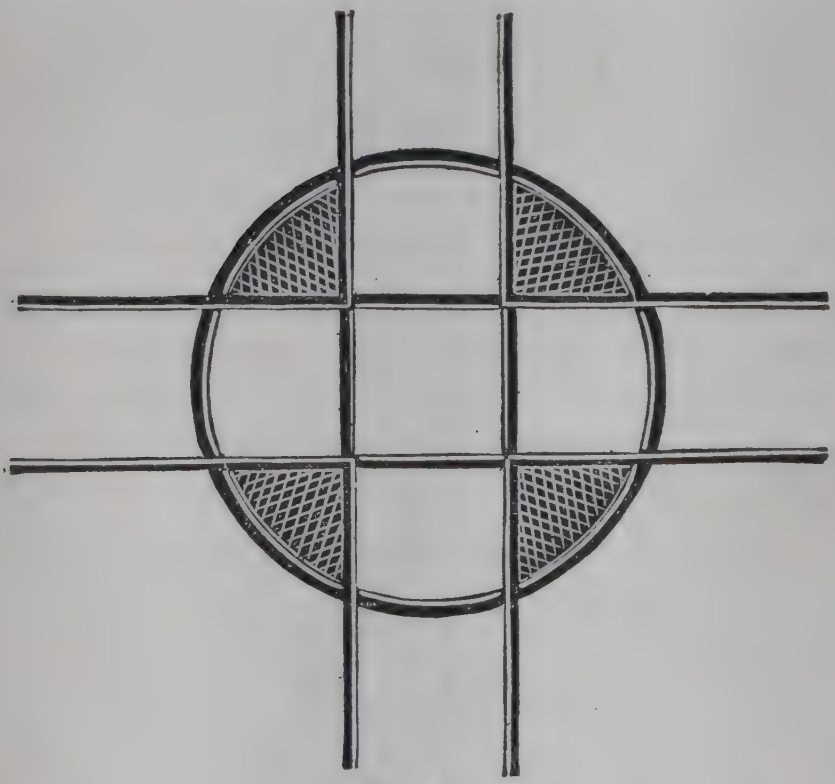


FIG. 171.

58. Three equal pencils of the same size and shape touch one another two by two. Find the volume of the interstice, if the diameter of each pencil be 4 mm.

59. The vertical section of a heap of iron rails 8 ft. long and 6 ft. high is an isosceles trapezium the parallel sides being 3 ft. and 1 ft. If there are only 200 rails in the heap and if the interstices form $\frac{1}{32}$ of the whole space, find the area of a cross section of a rail, its length being the same as that of the heap.

60. Two buoys A and B and a boat between them are in a straight line, the distances of A and B from the boat are 1·6 miles and ·9 miles. The boat moves in a direction at right angles to the line joining the buoys and in 15 minutes reaches a place at which the line joining the buoys subtends a right angle. Find the velocity of the boat.

61. Make rainfall graphs from the numbers in the subjoined table. Which are the months in each case when the rainfall is greatest? Which are the months in each case when the rainfall is above the average for the year?

Month.	Calicut.	Darjeeling.
	Rainfall in in.	Rainfall in in.
1	...	1
2	...	1
3	1	2
4	4	4
5	9	8
6	36	24
7	29	32
8	15	26
9	7	18
10	9	5
11	4	...
12	1	...

62. Prepare a temperature graph from the following table:

Month.	Temperature in Degrees F.		Month.	Temperature in Degrees F.	
	Montreal.	Port Darwin.		Montreal.	Port Darwin.
1	12	78	7	70	84
2	16	80	8	69	84
3	25	85	9	61	85
4	40	86	10	48	86
5	55	87	11	34	82
6	65	86	12	18	79

Mark the average temperature by a horizontal line in the figure.

63. From the following table, plot a graph showing the relation between the length of a river and the area of the river basin.

River.	Length in miles.	Area of basin in square miles.
Elbe ...	720	55,000
Rhine ...	810	87,000
Rhone ...	1,100	807,000
Danube ...	1,800	315,000
Volga ...	2,350	563,000
Thames ...	210	6,000
Severn ...	210	6,000
P. O. ...	420	27,000

64. From the following table, plot graphs showing the relation between area and population; and from the slopes of the graphs arrange and mark in the figure the countries according to the density per square mile. Verify your answer by calculatiug the density per square mile.

Country.	Area in 1000 square miles.	Population in millions.
Portugal ...	35	5
Spain ...	195	20
Italy ...	111	34
Greece ...	25	3
Servia ...	19	3
Bulgaria ...	38	4
Turkey ...	65	6

65. For a certain machine and for various weights the following table gives the power (P lbs) to be applied in order to move the weight (W lbs) :—

W	50	100	200	300	400	500	600
P	106	122	152	184	216	246	278

Draw a graph showing the relation of W and P, and assuming the relation between P and W to be of the form $P = a\,W + b$, find the values of a and b. From the graph, and also from the equation, find, to the nearest integer, the value of P when W = 350, and the value of W when P = 158.

66. The following table gives the meter reading at the end of a month which will indicate the number of units of electricity consumed. Fill up the blanks and draw a graph showing the number of units consumed in each month :

Date.	Meter reading.	Monthly consumption.
1-1-11	710	
31-1-'11	743	33
28-2-'11	775	32
31-3-'11	816	...
30-4-'11	860	...
31-5-'11	918	...
30-6-'11	1,005	...
31-7-'11	1,046	...
31-8-'11	1,084	...
30-9-'11	1,145	...
31-10-'11	1,184	

67. A truck full of cases the weight of a case varying from 3 qrs. 24 lbs. to 3 qrs. weighs 16 tons 13 cwt. 2 qrs. The empty truck weighs 8 tons 4 cwt. 2 qrs. Find the maximum and the minimum number of cases in the truck.

68. In the Secondary School Leaving Certificate of a Fifth Form pupil in a school, it is given that the Form average in Elementary Mathematics in the first term of 1910-11 is 41 per cent. Supposing there are 39 pupils in the Fifth Form in the school and that, out of the other 38 pupils.

9 pupils get 52 per cent. each
12.....42.....
10.....38.....
7.....32.....

find the percentage of the marks that the pupil gets.

69. A ruler supposed to be 6 inches long but which is actually 5·7 in. long is equally divided into 6 parts and each part again into 10 equal parts. A rectangle is measured by means of this ruler and found to be 3·8 in. by 2·9 in. Find the error in the calculated area of the rectangle.

70. On a board 4 ft. by 3 ft. a triangle is drawn and a circle is inscribed within it. The points of contact of the circle with the sides of the $\triangle$ are at distances of 1·2 ft., 1·9 ft. and 1·2 ft. respectively from the bottom left-hand corner and the straight lines

joining these points and the bottom left-hand corner make angles of 60°, 45° and 30° respectively with the bottom edge of the board. Draw a neat figure to scale and find the sides of the Δ by measurement.

71. The following is the advertisement of the New Encyclopedia Britannica (11th edition) by the Cambridge University Press: 4 monthly payments of Rs. 164-4 as., the first instalment payable 1 month after delivery. Cash price Rs. 652-8 as. At what rate is the simple interest calculated?

72. Fig. 172 is the plan of a room (not accurately drawn to

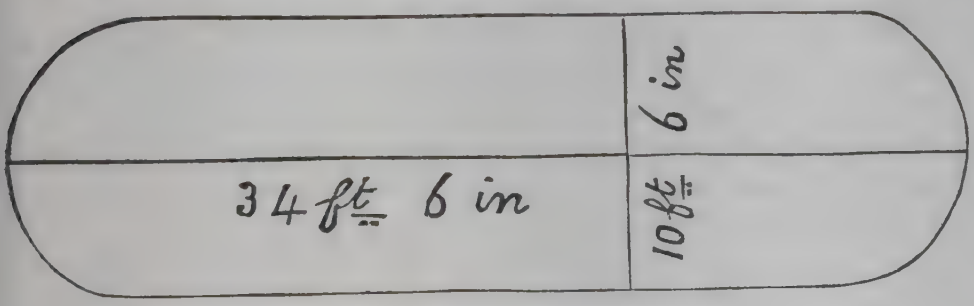


FIG. 172.

scale), ends terminating in semi-circles. If the wall all round is 12 ft. high and 1 ft. 9 in. thick, find the cost of painting the wall inside and outside at 4 as. 6p. per sq. yd., making allowance for two doorways each 6 ft. by 4 ft.

73. A train starts from Beach at 3·30 P.M. and goes south at the rate of 20 miles an hour. The Boat Mail starts from Beach at 5·35 P.M. and goes south at the rate of 30 miles an hour, supposing that the Boat Mail stops at the 36th mile 15 minutes and that the other train rests at the 3rd, 14th, 36th and 80th mile 10 min., 5 m., 15 m. and 5 m. respectively, find graphically when and where the Boat Mail overtakes the other train.

74. In the long term of a school there are 115 working days and the Headmaster has to express as a percentage the attendance put in by each boy. He calculates the percentage for any number of days by the help of the following table:

the percentage for 1 day out of 115 days = ·86856...
.....2 days.....= 1·73712
.....3.....= 2·60568
.....4.....= 3·47424
.....5.....= 4·34280
.....6.....= 5·21136
.....7.....= 6·07992
.....8.....= 6·94848
.....9.....= 7·81704

Using the table find the percentage for (1) 112; (2) 97; (3) 68 days.

75. The Ceylon Boat Mail leaves Chingleput at 5-53 and runs without stopping at intermediate stations to Egmore 37 miles distant, the time of arrival being 7·32. A local train leaves Pallavaram (a station between Egmore and Chingleput) for Egmore and travels according to the following time-table:

Miles from Egmore.		
11	Pallavaram	... 7—0
8	Mount	... 7—9
5	Saidapet	... 7—19
4	Mambalam	... 7—25
3	Kodambakam	... 7—30
1	Chetpet	... 7—37
	Egmore	... 7—44

At which station must the local train be shunted so as to allow the Mail train to go through without being stopped? Draw a graph showing the running of the two trains,

76. A person sends from India to England Rs. 780 per calendar month. If a rupee be worth 1*s.* 9½*d.* what will be the value in £ *s.* *d.* of the sum sent in a year? If a rupee be worth only 1*s.* 7½*d.*, how many rupees per month must be sent that the value in £ *s.* *d.* of the sum sent in a year may remain the same as before?

77. A farm consists of Nunja lands let at Rs. 30 per acre and Punja lands let at Rs. 20 per acre, the total rent being Rs. 940. When the rent of the Nunja is reduced by Rs. 5 per acre and that of the Punja by Rs. 10 per acre, the total rent is reduced by Rs. 290. Find how many acres of each kind there are in the farm.

78. The tax on an article having been increased from 5 per cent. to 7 per cent., the revenue derived from it was found to be diminished by one-third. Find the diminution in the consumption of the article.

79. (*a*) Taking $\pi = 3\frac{1}{7}$. Show that $\frac{1}{\pi} = \frac{1}{4} + \frac{1}{16} + \frac{1}{176}$. Use this to multiply 68·37 by $\frac{1}{\pi}$ to two places of decimals.

(*b*) Ptolemy gave the value of π by saying that $\pi^\circ = 3^\circ\ 8'\ 30''$. From this obtain his value of π as a decimal fraction (to four places of decimals).

80. "The volume of a sphere of diameter of d inches is ($\frac{1}{2}d^3 + \frac{1}{20}$ of $\frac{1}{2}\ d^3$) cu. in." Find the percentage error. To how many significant figures is this correct for a sphere of diameter 5 in.?

81. A sovereign may not weigh more than 123·47447 grains and has to be withdrawn from circulation when it weighs less than 122·5 grains. A heap of sovereign weighs 12 lb. troy to the nearest grain. State the greatest and least possible number in the heap.

82. A sum of money put out at compound interest amounts in 2 years to £7041-13-4 and in 3 years to £7323-6-8. What was the rate of interest and the principal?

83. Instead of paying Rs. 100 cash a man agrees to pay a certain sum at once and an equal sum after a year. How much ought each sum to be, reckoning interest at 4 per cent. per annum?

84. A man borrowed Rs. 350 at 4 per cent. After 3 months he borrowed anothor Rs. 100 at $4\frac{1}{2}$ per cent. and 4 months after the latter, he borrowed another Rs. 100 at 5 per cent. At the end of a year from the first loan he pays back the whole principal and interest. What sum could he have borrowed for the whole year at $4\frac{1}{2}$ per cent. interest so as to have to pay back the same sum at the end of the year as in the first case?

85. Which gives the more favourable terms to a customer, a bank paying 3 % per annum on deposits and making accounts half-yearly, or one paying $3\frac{1}{2}$ % per annum and making accounts yearly?

86. A tenant takes a house for three years, the rent is Rs. 2,400 a year and is due at the end of each year. If the rate of interest

is 5 per cent. per annum how much must the tenant pay down at once to reduce his yearly payment from Rs. 2,400 to Rs. 2,000 ? [Reckon compound interest.]

87. A person owned property yielding a rental of Rs. 1,750 per mensem of which he had to spend 37½ per cent. on repairs, management, taxes, etc. He sold the property realising Rs. 3,65,750 which he invested in 3½ per cent. Government paper at 96¼. If he has to pay income-tax at the rate of 5 pies in the rupee on the interest derived from the investment, find the alteration in his clear annual income. [Mad. Mat. 1900.

88. A person invested Rs. 16,500 in 3½ % Government paper at 96¼ and an equal sum in bank shares of the nominal value of Rs. 500 each. If, when the bank is paying dividend at the rate of Rs. 50 per share, his annual income from the bank shares exceeds his annual income from the Government paper by Rs. 87-8-0, find what he paid for each of his bank shares. [Mad. Mat. 1898.

89. A person sold 25 Bank of Madras shares and invested the proceeds in the Government 3½ per cents. when they were at 3¼ premium. If his net annual income from the investment after paying income-tax at the rate of 5 pies in the rupee be Rs. 876-9-0, find the price at which he sold each of his Bank shares. [Mad. Mat. 1894.

90. The capital of a Railway Company amounts to Rs 18,90,00,000 of which one-fourth is 5 per cent. preference stock and one-third 4½ % preference stock. In a certain year the receipts are 1,81,50,000 and the working expenses amount to 55 per cent. of the receipts. Of the net receipts Rs. 5,40,100 are added to the reserve fund and the remainder after paying dividend on the preference stock is divided among the ordinary shareholders. What rate of interest will they receive ?

91. A merchant in Madras owing £ 398-5*s*. 9*d*. to a merchant in London, finds that by remitting through Paris instead of direct he can save Rs. 58-8 as. If exchange between Madras and Paris is at the rate of 1·71 francs per rupee and exchange between Paris and London at the rate of 25·2 francs per £ 1, find the rate of exchange between Madras and London. [Mad. Mat. 1900.

92. A merchant in Madras owes 12,270 marks to a merchant in Hamburg. If exchange in Hamburg is at the rate of 1·32 marks per rupee while exchange in London is at the rate of 1*s*. $3\frac{3}{4}d$. per rupee and the exchange between London and Hamburg is 20·45 marks per pound sterling, find to the nearest pie how much the merchant will gain by remitting through London instead of direct.

[Mad. Mat. 1897.

93. A merchant at Madras imports 600 tons of English coal the price of which at the pits' mouth is 1*s*. 6*d*. per ton. The cost of carriage and freight to Madras amounts to 16*s*. $6\frac{1}{2}d$. per ton and landing charges amount to Rs. 3-5-4 per ton. If the merchant remits the price of the coal and the cost of carriage and freight to Madras when exchange is at the rate of 1*s*·$3\frac{15}{16}d$. per rupee, at what rate per maund of $82\frac{4}{7}$ lbs. avoirdupois must he sell the coal in Madras in order that he may gain Rs. 3,255 on the transaction?

94. A Parsee Theatrical Company chooses to stay at a place from Wednesday, the 1st March to Wednesday, the 31st May 1911 and enters into an agreement with a contractor that he should pay the Company a sum of Rs. 1,50,000 and it should enact in return one play every night except on Sunday when there will be a matinee performance (Friday being a holiday). The Company realises an income of Rs. 4,000 for every matinee, Rs. 5,000 every Saturday night and an average income of Rs. 2,500 every other night. If all incidental charges come to Rs. 4,000, find whether the contractor gains or loses and by how much per cent.

95. It is proposed by the Manager of a school to introduce the provident fund system for the benefit of the teachers in his service. There are 15 members of the staff, one of whom is paid Rs. 150, two Rs. 100, three Rs. 80 and the rest on an average of Rs. 40. Each teacher should pay one anna in the rupee of his salary, and the Manager contributes an equal sum. The Manager wants to throw the burden on boys and enhances their fees proportionately. If the strength of the school is 600, by how much will the fee of each boy on an average be increased?

96. A person who takes 10 shares in the Mylapore Hindu

Fund Limited has to pay Rs. 10 every month to the Fund. At the end of 84 months the Fund pays him Rs. 1,000. At what rate is simple interest charged? Given that $1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2}$.

97. Sketch the plan and calculate the area of a field ABEGFDC from the following notes :—

	yds. To G	
	204	
To F84	188	
	120	15 to E
To D65	115	
To C15	84	
	60	70 to B.
	From A	

98. Shew that the straight lines represented by the equation $2x + 7y = 25$; $3x + 11y = 39$; $8y - 7x = 10$; pass through the same point.

99. Figure (173) is drawn on a scale of 20 ft. to the inch and

FIG. 173.

represents the end view of a shed 40 ft. long, the roof and side

walls of which are made of corrugated iron. Find the cost of the roof and side walls at 8 as. per sq. foot.

100. By pricking off Fig. 174 into your book and counting squares, find the area in sq. centimetres of the figure enclosed in.

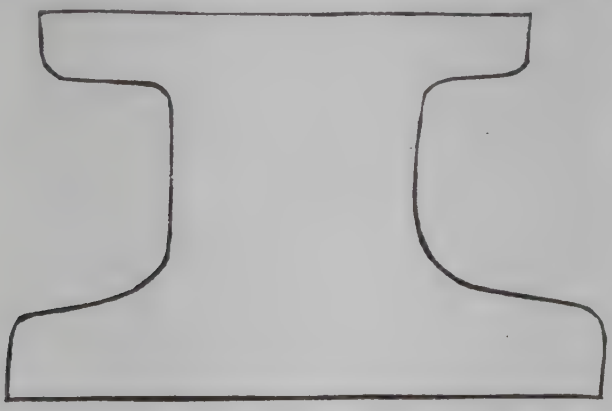


FIG. 174.

Find the weight of iron rail of this section for a single line railway 20 km. long. 1 cub. cm. of iron weighs 88 grammes.

101. ABC represents the circumference of a new grindstone weighing 104 lbs. Find its weight when it is worn down to DEF. Ignore the hole through the stone.

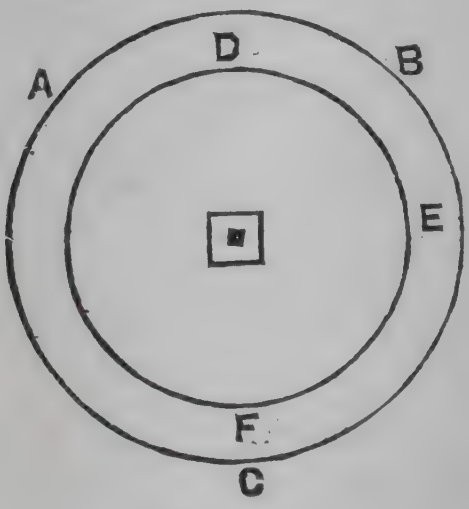


FIG. 175.

102. The highest and lowest marks gained in an examination are 180 aud 40. The marks are to be reduced so that 180 be-

comes 120 and 40 becomes 30. Draw a graph and from it find the marks to the nearest integer to be assigned to papers that obtained (1) 142 marks (2) 83 marks.

103. Find the area of the property whose plan on the scale of 2 chains to 1 inch is given in Fig. 176.

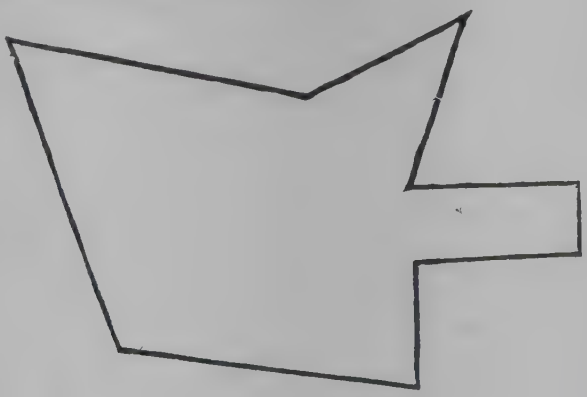


FIG. 176.

104. A square field whose area is 115·7776 sq. yds. is to be enclosed with barbed wire placed at heights of 1, 2, 3 and 4 feet above the ground. Find the length of wire required, to the nearest yard, if the length of wire required for each circuit is 4 $^{0}/_{0}$ greater than the perimeter of the field.

105. The distance at which the top of the mast of a ship can just be seen from a point in the sea level is proportional to the square root of the height of the mast. If a mast 60 ft. high can be seen at a distance of 10 miles, find (*a*) the distance at which a mast 24·35 ft. high can be seen; (*b*) the height of the mast of a ship which can just be seen at a distance of 6 miles.

106. A triangulur field ABC is to be levelled. The side BC is level and A is 3·4 metres higher than BC. Find by how much the field is to be lowered at A and by how much it is to be raised along BC.

107. Find the cubical contents of the wood in the rim of a wheel shown in the following figure(scale $\frac{1}{50}$), thickness being 1 in.

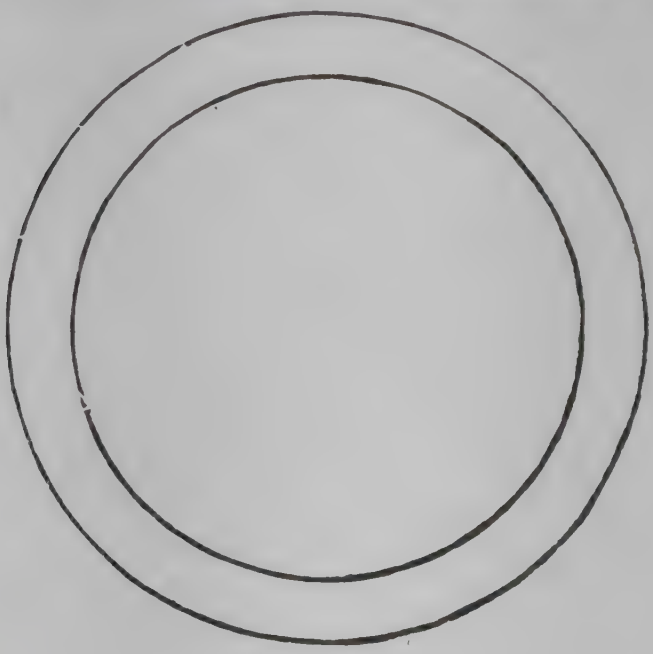


FIG. 177.

108. A glass tumbler of the shape and size shown in Fig. 178

FIG. 178.

(scale $\frac{1}{4}$) is filled with water to half the depth and the rest with oil. Find the quantity of water and that of oil contained in it.

Madras Secondary School Leaving Certificate.

Public Examination.

1911.

[*Only* **ten** *questions to be attempted.*]

1. Find graphically what the remainder is when 189 is divided by 56 by factors.

2. A rectangle measures 10·05 cm. by 9·93 cm. ; find the area of the rectangle without directly multiplying these two numbers together.

3. Find the cost, to the nearest anna, of a school library as follows :—

No. of Books.	Published price of each.	
	s.	*d.*
128	7	6
200	5	0
20	3	6
236	3	0
256	2	6
104	2	6 net.
350	2	0
220	1	0 net.

(*a*) 12 as. to the shilling.

(*b*) $6\frac{1}{4}$% discount allowed off the published prices, except for net books.

(*c*) Packing and carriage 2% of the published prices of the books.

4. The following are the results of a recent examination :—

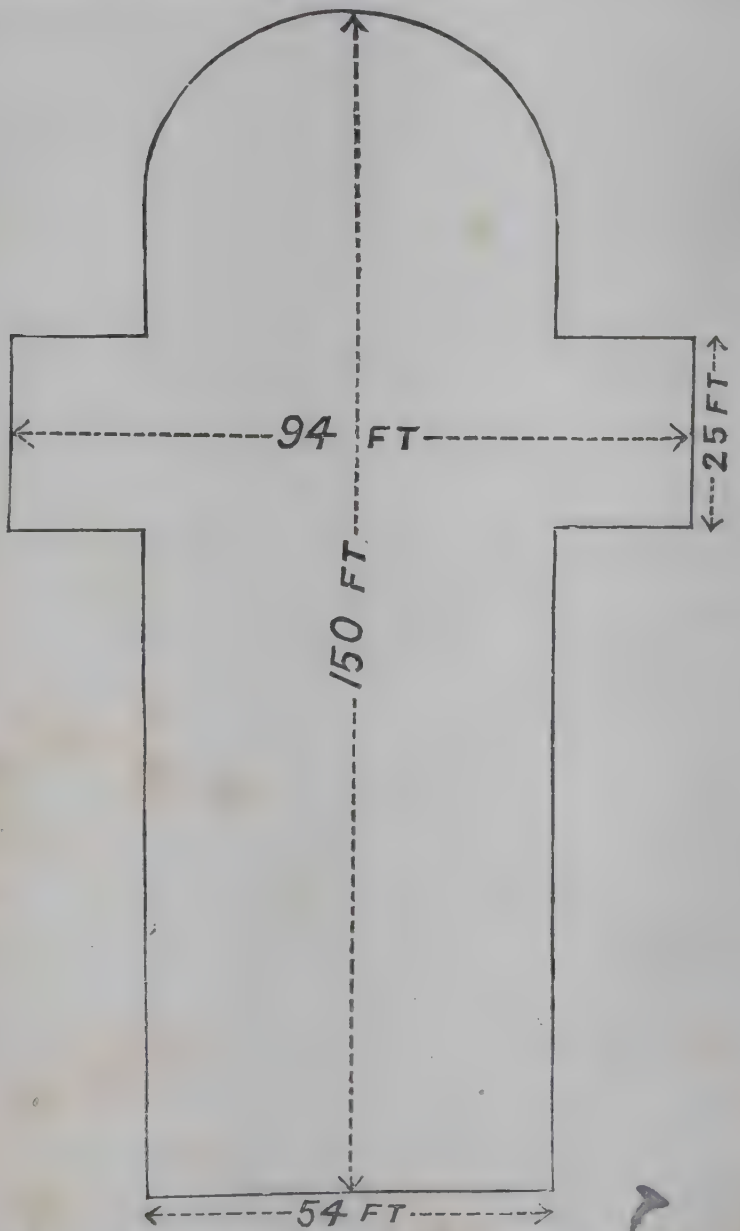
Centre	No. of Candidates.	No. of Passes.
A.	24	10
B.	119	60
C.	327	106
D.	218	49
E.	196	37
F.	623	214
G.	257	98
H.	69	44

Find, correct to ·1, the percentage of passes for the whole examination ; and give the percentage of passes at the most and at the least successful centres.

5. I bought a house for Rs. 6500 ten years ago. I charge Rs. 40 a month for rent, and pay the house tax (2*s.* 8*d.* a month, and do the repairs (averaging Rs. 60 a year) myself. It has been let for an average of $9\frac{1}{2}$ months in every year, and while it stood empty I had to pay a man Rs. 5 a month to look after it. At the

same time a friend of mine deposited Rs. 6500 in a bank and has drawn annually interest at the rate of $4\frac{1}{2}\%$ per annum. Which of us has benefited the more from his investment, and by how much?

6. This is the plan of a lecture-hall (not drawn accurately to scale), one end terminating in a semi-circle.



Find the area of the floor. The average height is 30 ft.; each

person requires for health 200 c. ft. of fresh air every hour. If a lecture lasts $1\frac{1}{2}$ hours and the air in the hall is not renewed, how many people can be accommodated without injury to health ?

7. At some School Sports the mile race is run on a perfectly circular course, the track being 8 yards wide. Four times round the course, on the inner edge of the track, is an exact mile. What is the length of the outer edge of the track ?

8. A boy uses a badly divided ruler supposed to be six inches long and to be divided into inches. As a matter of fact it is so divided that the first, second, &c., so-called inches are ·98, 1·03, 1·01, ·96, ·94, and 1·01 inches (each 'inch' being divided into 10 equal parts). He measures a line and finds it, according to his ruler, 3 ft. 4·6 in. What is the real length of the line ?

9. A boy hires a bicycle by paying one rupee for the use of the bicycle plus eight annas for every mile that he rides :—

(1) Express the cost of hiring in terms of the number of miles ridden.

(2) Draw a graph representing the relation between the cost and the distance.

(3) From the graph find how many miles he can ride for Rs. 5.

10. The distance from Madras to Tuticorin is 440 miles. The following trains run :—

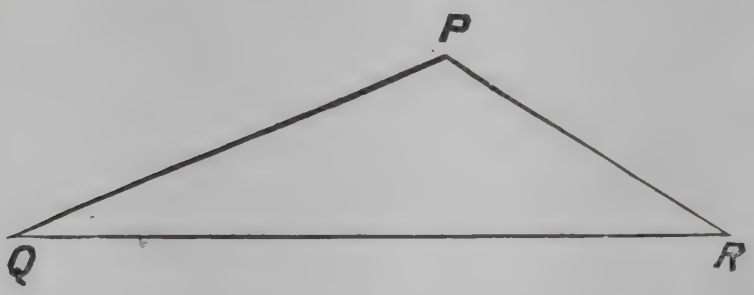
(1) Monday, Madras dep. 6 p.m.
Tuesday, Tuticorin arr. 4 p.m.
(2) Monday, Tuticorin dep. 8 p.m.
Tuesday, Madras arr. 6 a.m.
(3) Monday, Tuticorin dep. 11 a.m.
Tuesday, Madras arr. 7 p. m.

Represent these three journeys graphically in one diagram. With no help but the diagram say (*a* how far each train is from Madras at 11 o'clock on Monday night ; (*b*) at what time the train from Madras (No. 1) passes the trains (Nos. 2 and 3) from Tuticorin. Omitting stoppages and assuming each train to travel uniformly.

11. A ——————————————— B

(1) Draw a line exactly equal in length to AB.
(2) Draw another line $\sqrt{3}$ times the length of AB.

12. Measure, as accurately as you can, each of the angles and each of the sides of PQR, and find its area.



13. On a blackboard measuring 4 ft. by 3 ft. is drawn a triangle so that :—(1) one vertex is 18 in. from the top left corner of the blackboard, and 24 in from the bottom left corner. (2) Another vertex is 24 in. from the top right corner and 18 in. from the bottom right corner. (3) The third vertex is 24 in. from the bottom right corner and the angle at this vertex is a right angle. Draw a plan showing the position of the triangle relative to the sides of the blackboard. State your scale below the plan. What is the length of the hypotenuse (or side opposite the right angle) ?

14. A goat is tied by a rope to a post 15 ft. from a long straight fence. The rope is such that the goat can reach with its mouth 23 ft. from the post. Along what length of the fence can it nibble ?

Draw to scale, choosing and stating your scale.

15. The full pension granted to an officer retiring from Government service is half his average salary for the last three years of his service ; but in the case of an officer who enters service after the age of 30, the full pension is reduced by one-fortieth for every year by which his age at admission into service exceeds 30. Express the pension of such an officer in terms of a full pension and of his age at admission into service.

Determine what pension a man who enters service at the age of 35 receives upon retiring, if his salary per mensem during his last 3 years of service is Rs. 375 for 10 mos., Rs. 350 for 3 mos., Rs. 370 for 8 mos., Rs. 380 for 9 mos., and Rs. 400 for 6 mos

1912.

[*Only* **ten** *questions to be attempted.*]

1.

	Elementary Mathematics.										Total.	%	Rank.
Max.	20	10	6	16	20	22	14	10	14	18	150		
A ...	14	8	3	13	13	16	12	4	10	12			
B ...	6	1	1	9	5	7	8	2	6	5			
C ...	2	2	0	6	1	4	5	1	3	2			
D ...	20	10	6	16	19	22	14	7	13	18			
E ...	13	3	2	12	12	14	11	3	9	11			
F ...	4	4	0	7	3	6	6	1	4	3			
G ...	1	5	0	5	0	2	4	1	2	1			
H ...	18	9	5	15	16	19	14	6	12	17			
L ...	*a*	4	1	11	*a*	11	10	*a*	8	8			
M ...	5	3	0	8	4	7	7	1	5	5			
N ...	7	2	1	10	6	8	9	2	7	6			
O ...	16	7	4	14	14	17	13	5	11	13			

In a table of 4 columns give—(1) the names (*A*, *B*, &c.) of these boys in alphabetical order; (2) the total of each boy's marks; (3) his percentage (in whole numbers) of the maximum; (4) his rank (determined by his percentage).

In case of absence the mark in col. 3 should be the boy's percentage of those maxima only for which he was present.

2. To celebrate the accession of Edward VII a certain town subscribed a sum of money which was spent as follows:—a proces-

sion, Rs. 57 ; feeding the poor, Rs. 214 ; fireworks, &c., Rs. 136 ; sports, Rs. 143 ; entertaining children, Rs. 184 ; local hospital, Rs. 250 ; office expenses, Rs. 36. If Rs. 1,445 had been subscribed last year and spent upon the same objects in the same proportion, how much (to the nearest anna) would have been spent upon each ?

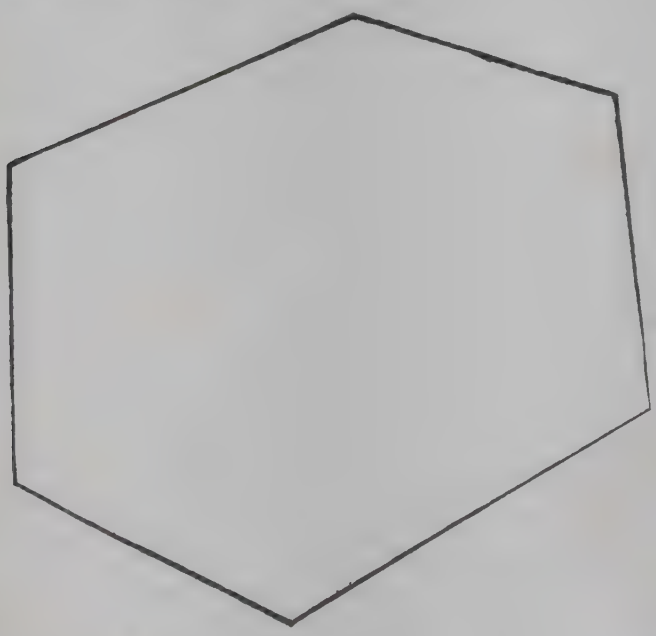
3. (1) A man invests Rs. 6,980 in $3\frac{1}{2}\%$ Government securities at 89 ; find his annual income.

(2) Find the simple interest for 9 months and 20 days on Rs. 1,264 at 10 as. per Rs. 100 per mensem.

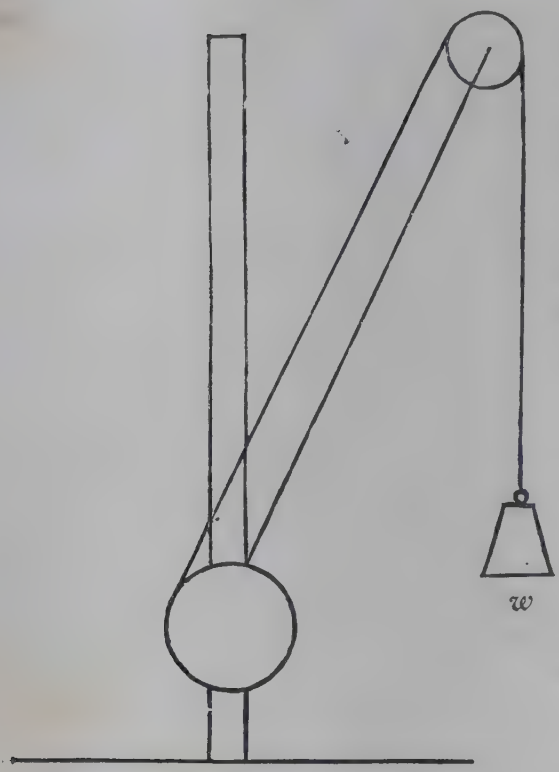
4. Allowing an average of 23 inches for each boy to stand in, if 198 boys are arranged in a circle, what will be its radius ?

If they are arranged in a rectangle of which each long side is double each short side, what will be the length of its diagonal in feet, correct to five significant figures ?

5. If this figure is an accurate plan (on a scale of $\frac{1}{10}$ in. to



50 yds.) of the boundary of an estate, find the area of the estate in sq. yds



6. This figure is intended to represent a crane, the arm of which can be completely revolved around the central post. The bottom of the weight (w) can be made to touch the ground and can also be raised 20 ft. above the ground. The distance of the vertical rope from the central post is 12 ft. and the radius of the central post is 6 in. If the weight is moved into every possible position by the crane, find the area of the surface over which a point on the bottom of the weight and vertically under the rope moves.

7. Draw a line 4·6 cm. in length and upon it describe an isosceles triangle having the angle at the vertex 30°.

Draw another triangle of the same form, the sides of which are to the corresponding sides of the first triangle as 7 is to 3.

8. The temperature of a patient taken every three hours was found to vary as follows :—

Monday,	4 p.m.	...	100·2°
„	7 „	...	101·4°
„	10 „	...	102·8°
Tuesday,	1 a.m.	...	103·6°
„	4 „	...	102·7°
„	7 „	...	100·5°
„	10 „	...	100°
„	1 p.m.	...	100·1°
„	4 „	...	100·3°

Draw on squared paper a curve indicating these changes; from it deduce what was probably his temperature at midnight on Monday and at 8-30 on Tuesday morning.

9. If a certain substance weighs 4 tolas for every cubic inch, find the weight of a solid right circular cone of the substance, of which the height is 20 in. and the diameter of the base 14·6 in.

10. Draw a circle of 3·3 cm. radius ; insert in it a chord of 4·2 cm. length, and bisect the smaller arc of the circle.

11. Construct a graph showing the relation between so many rupees a month and its equivalent in pounds sterling a year. From it deduce how many pounds a year are equivalent to Rs. 40 a month ; and also how many rupees a month are equivalent to £60 a year.

12. If a metre = 39·37 in. and a kilogram = 2·2 pounds, express a pressure of so many pounds per sq. ft. in terms of its equivalent in grammes per sq. cm.

13. The estimated cost of an irrigation project is 2 lakhs, and it is expected to yield, after deducting for maintenance charges, a net return of 5·2%. Without such deduction, however, the return would be 6·4%. Assuming the maintenance charges to be 12 as. per acre, find (1) the extent of land proposed to be irrigated, and (2) the water-rate per acre proposed to be levied.

14. The time (t) in seconds of an oscillation of a pendulum is given by the formula $t=\pi\sqrt{\frac{l}{g}}$ where l is the length of the pendulum. Find the length in cm. of a pendulum which oscillates 89 times in a minute.

$$g=981\ ;\ \pi=3{\cdot}142.$$

15. The income and expenditure of a certain school are as follows :—school fees, Rs. 12,672-12 ; fines, Rs. 9-7 ; stationery fees Rs. 226-8 ; library fees, Rs. 318-6 ; laboratory fees, Rs. 116-8 : entrance fees, Rs. 56 ; miscellaneous receipts, Rs. 27-7. Salaries, Rs. 12,028-8 ; contingencies, Rs. 125-4 ; stationery, &c., Rs. 314-8 ; library, Rs. 367-3 ; class books, Rs. 63-8 ; furniture, Rs. 281-5 ; magazines, &c., Rs. 213-6 ; laboratory, Rs. 187-8 ; prizes, Rs. 129-3 ; annual repairs, Rs. 253-7.

If Government gives a grant of half the deficit, find the cost to the proprietors per boy, there being 540 boys.

1913.

[*Only* **nine** *questions to be attempted.*]

1. (1)

Cities.	*Population in* 1911.	*No. of females to* 1,000 *males.*
Madras ...	518,660	946
Madura ...	134,130	999
Trichinopoly ...	123,512	1,006
Calicut ...	78,417	928
Kumbakonam...	64,647	1,064

From the above table find (i) the total population of these five towns, and (ii) the number of males in each town.

(2) Find, correct to three significant figures,

$$0{\cdot}0923 \times 13{\cdot}215 \times 2{\cdot}654364.$$

2. (1) What is Rs. 698-12-0 in English money if exchange is at 1*s.* $4\frac{5}{16}$*d.* to the rupee?

(2) A man placed Rs. 1,500 as a fixed deposit in a bank paying 6 % interest, the interest being added to the capital every year. Four years afterwards the bank failed and paid a dividend of two annas in the rupee. What did he get from the bank?

3. If a metre is 39·3708 inches, find, correct to the nearest foot, the difference between 83 kilometres and 51 miles.

4. Solve the following equations algebraically, and check the result by means of a graph:

$$3x - \tfrac{2}{3}y = 10; \quad \tfrac{3}{4}x - 1\tfrac{1}{3}y + 1 = 0.$$

5. In a certain bicycle the hind wheel revolves two and a half times for one revolution of a pedal. If the outside diameter of the hind wheel is 28 inches, express the distance travelled in terms of the number of revolutions of a pedal; and find the distance when the number of revolutions is 500.

6. (1) Of two cylindrical vessels the diameter of the second is double that of the first, and the height of the second is half that of the first. Compare the capacity of the second with that of the first.

(2) If each has some water in it and a stone is immersed first in the one and then in the other, compare the height by which the water would rise in the second with the height by which it would rise in the first.

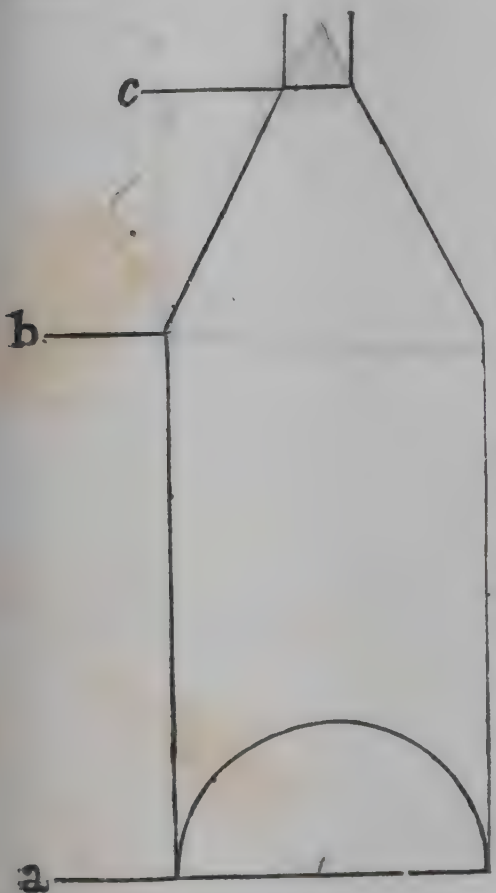
7. (1) Draw four parallel straight lines such that the distance

between the first and second is 1 in., between the second and third $\sqrt{2}$ in., and between the third and fourth $\sqrt{3}$ in., finding $\sqrt{2}$ in. and $\sqrt{3}$ in. geometrically.

(2) Draw a straight line cutting these four parallels and making an angle of 45° with them. What length of this line is intercepted between the second and third parallels?

8. (1) Draw a triangle whose sides are 1·8, 2·4, and 3 in. respectively; give the measurement of each angle, and state what is the area of the circumscribing circle.

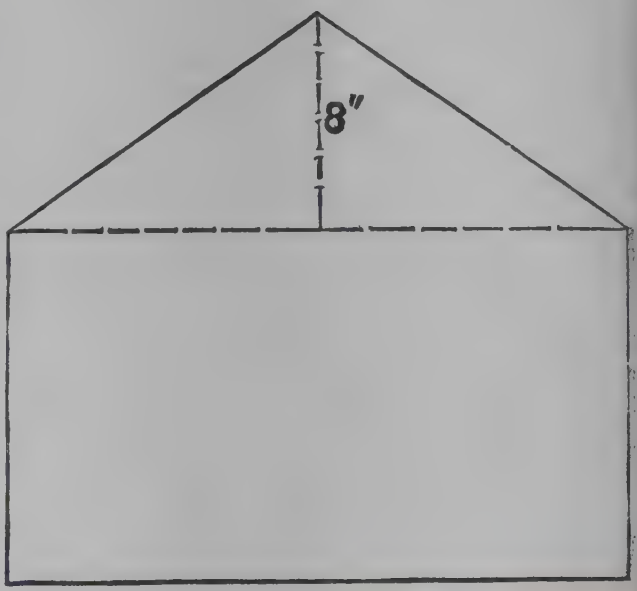
(2) Draw a right-angled triangle and draw the square on each side; then show, by dotted lines and by instructions given below your figure, how to cut up the squares on two sides and, by superposition of the cut pieces upon the third square, how to prove that the area of two of the squares is equal to that of the third square.



9. The figure represents a bottle, cylindrical in shape, with a hemispherical bottom thrust in as shown, and with a shoulder which is the frustum of a cone; diameter of the cylindrical portion 6 in., diameter of the bottom of the neck (or top of the shoulder) 2 in., height from *a* to *b* 12 in., and height from *b* to *c* 5 in. Find the capacity of the bottle up to the bottom of the neck.

10. For every acre of wet land a man pays a land revenue tax of Rs 8; and for every rupee of land revenue he pays a local cess of 1 a. 6 p. Draw a graph showing the total tax paid upon land varying in extent from 20 to 40 acres. What would be the total tax on 35 acres?

11. Find the cost of whitewashing a room which is 40′ by 18′ with walls 14′ high, allowing for *half* the area of the following openings: two doorways, each 7′ by $4\frac{1}{2}$′; four windows, each $3\frac{1}{2}$′ by $4\frac{1}{3}$′; and two circular ventilators, each 2′ in diameter. The wall at each end of the room is as in the figure.



Whitewashing: 2 a. 6 p. per 100 sq. ft.

12. An open rectangular cistern, of which the internal length, breadth, and depth are as 4 : 3 : 2, is made of metal 1·5 cm. thick and 8 times as heavy as water. If the capacity of the cistern is 12,288 litres, what is its weight in kilograms when empty?

13. A bookseller is allowed by a publisher 20 °/₀ discount on the nominal price of a book and 5 °/₀ on this last amount for cash. For every dozen books he gets an extra copy for nothing. The bookseller allows his cash customer 4 annas in the rupee off the nominal price. Find (to one decimal place) the bookseller's profit on cash sales.

ANSWERS.

Page 4.—*Oral.*

1. $18a$. 2. $-15a$. 3. $7b$. 4. $-4a$. 5. $-\frac{3a}{2}$.
6. $4b$. 7. $-6ab$. 8. $-48cd$. 9. $12bc$.
10. $7{\cdot}5xy$. 11. $-72{\cdot}lm$. 12. $18abe$. 13. $60abc$.
14. $-144abc$. 15. $-\frac{1}{6}abc$.

Page 5.—*Oral.*

1. 12. 2. -12. 3. 3. 4. -2. 5. -7.
6. $-6\frac{2}{3}$. 7. 9. 8. 5. 9. $-9a$. 10. 4.
11. $-4b$. 12. $2c$. 13. a. 14. $\frac{6}{25}y$. 15. $\frac{b}{5}$.
16. $2c$. 17. $-2xy$. 18. $5c$.

Exercise I (a). p. 7.

1. a^3b. 2. $-36x^4$. 3. $56a^2b^3c$.
4. $-6a^4b^3c^3$. 5. $-315a^7b^4c^2d^2$. 6. $-\frac{1}{10}a^4bc^5d$.
7. $\frac{7}{36}l^3m^5n^3pq$. 8. $64a^6b^4c^2$. 9. $\frac{5}{36}b^6c^2d^4z^2$.
10. $-512a^3b^9c^3$. 11. $\frac{1}{729}b^9d^3e^3$. 12. $\cdot001953125a^9c^3$.
13. $192a^4b^4c^3dxy$. 14. $10bx^6y^2$.

(1) 4. (2) 0. (3) -1323. (4) 147456.

Exercise I (b). p. 9.

1. a^4. 2. $-7x$. 3. xy.
4. $-7abc^2$. 5. $-a$. 6. $3x^{m-p}$.
7. $\frac{9}{4}c^2$. 8. $-8x^{m-p}y^{n-p}$. 9. $a^mb^nc^{2p}$.
10. $-2lmn$. 11. $\frac{16}{3}$. 12. $abcd^3$.
13. $\frac{18c}{35b}$. 14. $\frac{24x^4}{49y^3z^{3a}}$. 15. $\frac{9}{7}$.
16. $72\frac{z^2}{xy}$.

Exercise I (c). p. 10.

1. $6ab + 30b^2$.
2. $56ac + 24bc - 16c^2$.
3. $6x^4 - 18x^3 + 24x^2 - 6x$.
4. $4a^5 - 36a^4b + 24a$.
5. $12a^3b + 18a^3c - 30a^2bc$.
6. $-7a^5b + 21a^4b^2 - 42a^4b + 28a^2b^2$.
7. $4ac + 42bc - 63c^2$.
8. $3x^2 - 10x + 5$.
9. $2xy - 2yz$.
10. $-3x^3 - 10x^2 - 15x$.

Exercise I (d). p. 11.

1. $3 - 4x + 5x^2$
2. $a^2 - 3ab + 3b^2$.
3. $2a - 3a^2c + 5ac$.
4. $a^2b^3 - 3ab^4$.
5. $7a - 21ab$.
6. $9x^3 - y^3$.
7. $xy + yz + xz$.
8. $a - 2b + 4c$.
9. $-4b^3 - 4c$.
10. $a - b$.

Exercise I (e). p. 12.

1. $a^5 + b^5$.
2. $a^4 + a^2b^2 - a^4b^2 + b^6$.
3. $2x^3 - 5x^2 - 8x + 15$.
4. $36x^3 + 71x^2 - 58x - 16$.
5. $a^4 + a^2b^2 + b^4$.
6. $9a^2 - 4b^2 - 36bc - 81c^2$.
7. $4x^6 - 9x^4y^2 + 14x^3y^3 - 9x^2y^4 + y^6$.
8. $x^2 - y^2 + x + 3y - 2$.
9. $4x^2 - 10x + 25$.
10. $4a^2 + 12ab + 9b^2$.
11. $9a^2 + 16b^2 + 81c^2 + 24ab - 72bc - 54ac$.
12. $16c^2 + 9b^2 + 4d^2 + a^2 - 24bc + 16cd - 8ac - 12bd + 6ab - 4ad$.

Exercise I (f). p. 13.

1. Quotient $x^2 - x + 1$. Remainder 0.
2. $x^2 + x + 1$; 0.
3. $x^4 - x^3a + x^2a^2 - xa^3 + a^4$; 0,
4. $x + 3y$; 0,
5. $a^2 + b^2 + c^2 - ab - ac - bc$; 0.
6. $x^2 - 1$; 0.
7. $x + 2$; 0.
8. $x^2 + 8x + 3$; $-25x - 51$.
9. $x^2 + 9x + 13 + \dfrac{6x - 64}{x^2 - 3x + 5}$.
10. $a^2 + 2a + 1 + \dfrac{2a + 6}{a^2 - 1}$.
11. $x + 5x$.
12. $x + 1$.

Exercise I (g). p. 14.

1. $2a^2b - 6ab^2$; -20.
2. $-50x^2 + 42xy - 61y^2$.
4. -1.
5. $ax^2 - ax + bx + a$.
7. (*a*) 121. (*b*) 49. (*c*) 98. (*d*) -36. (*e*) 104 : 316 ; 16584. (*f*) 53 ; 8 ; 143.
8. 0 ; 32.
9. (*a*) $x^3 + 6x^2 + 11x + 6$.

(*b*) $x^2y - xy^2 + y^2z - yz^2 + z^2x - zx^2$. (*c*) $4x^4 - 5x^2 + 1$.

10. $63x^3$. 11. $8a^3 - 17a^2 - 11a + 3$ ft.; $\frac{8a^3 - 17a^2 - 11a + 3}{4}$ ft.

12. $£(5x^3 - 4x^2 + x)$; $\frac{4x^3 + x^2 - x + 5}{5x^3 - 4x^2 + x}$.

13. $(2x^5 - 11x^4 + 20x^3 - 17x^2 - 8x + 12)$ sq. yds.;
Rs. $\frac{3y(2x^5 - 11x^4 + 20x^3 - 17x^2 - 8x + 12)}{64}$.

14. $(4x + 3)$ miles per hour; $x = \frac{1}{4}$.

15. $\frac{1}{2}(pa + pb + qa + qb)$ square units. 16. $(a^4 - 1)$ sq. ft.

17. Yes. 18. 66. 19. (i) 16. (ii) 12. (iii) 0.

20. (1) -24. (2) 0. 21. Rs. $\frac{h(100 - l)(100 + p)}{100 \times 100}$.

22. ab sq. ft.; $\frac{576v^2 + 96v(3a + 3b + c)}{sd}$ slabs.

23. $435600c(a + b - c)$ sq. ft; $435600(ab - ac - bc + a^2)$ sq. ft.

24. 84 kg. $\left\{\left(\frac{1000p}{w} - \frac{c}{g}\right) \times g\right\}$ kg.

25. $\frac{wlhe}{2240}$ tons, $\frac{a}{wh}$ ft. per minute; $64\frac{2}{7}$ tons, $33\frac{1}{3}$ ft. per minute.

26. 9·83 inches, 3·22 inches.

27. 26188·8 lm^2 gallons; $\frac{ng}{314265·6}$ miles per hour.

28. (*a*) $1 + x + x^2 + x^3$; 1·010101 } 1·01 } the results are correct to
(*b*) $1 - x + x^2 - x^3$; ·990099 } ·99 } two decimal places.

29. 3·16 inches nearly; 3·04 m. nearly.

30. 16500 ft. above the sea level.

31. 5300 ft. below the sea level.

32. $\frac{55br}{54d}$ miles per hour; $\frac{11}{45}$ miles per hour.

Exercise II (a). p. 22.

2. ·564 of a mile nearly. 3. 3·46 ft.

4. CD = 2·3 cm.; CE = 3·7 (cm.) nearly DE = 1·7 cm. nearly.

5. They are similar. 8. 10 and 15 ft.

12. sine 60° = ·86, cos 60 = ·5, tan 60 = 1·79.

Exercise II (c). p. 27.

2. ·69″ ·89″. 5, 1·87″, 2·26″, 3·23″. 6. ·9″, 1·3″, 1·7″, 2·2″, 2·6″.

8. 3 yds. 2 ft. 1 in., 2 yds., 1 ft. 6 in.

Exercise II (d). p. 29.

1. (1) $\frac{1}{36}$. (2) $\frac{1}{6336}$. (3) $\frac{1}{79200}$.
2. 58 yds. 3. $\frac{1}{6336}$, 2·27″. 4. $\frac{1}{100000}$, 1·27.

Exercise II (e). p 31.

1. 128, 108. 2. 16, 18. 3. 1·39, 5·45; 3·73.
4. (*a*) ·29. (*b*) ·02. (*c*) 4·28. (*d*) 2·61.
5. (*a*) 3·67. (*b*) 1·95. (*c*) 1·76. (*d*) 18·51.

Exercise II (f). p. 35.

6. $\frac{60}{7}$, $\frac{180}{7}$, $\frac{300}{7}$, 60, $\frac{540}{7}$, $\frac{660}{7}$, $\frac{780}{7}$. 73·29 metres = DC, B = 1·95 m.
12. $\frac{9}{10}$ of an acre. 13. $\frac{36}{67}$. 14. 1456 sq. yds.; $\frac{9}{67}$

Exercise II (g). p. 37.

1. 4·6″, 9·4″. 2. $\frac{10881}{12225}$ lbs.
3. sin 15° = ·3. cos 15° = ·95. tan 15° ·31. 4. 1·72
5. 1837·5. 6. 6 ft. 7. 18 ft., 4·5 in.
8. 22 ft. $6\frac{3}{4}$ ft. 10. A circle. 11. $41\frac{9}{11}$ of an acre.
13. 12 ft.; 37 or 38 stones. 14. 8 s. in. = ·5, cos = ·8.
15. 2·2 ft. 16. sin = ·512. cos = ·8. 17. 60 yds.
20. Cannot be seen. 21. 98000 sq. miles. 22. $\frac{1}{24665}$
24. 330 W. 200 to the North.

Exercise III (a). p. 42.

1. $a\,(a+b)$. 2. $b\,(b+2c)$. 3. $p\,(p+2q+2r)$.
4. $a\,(ab+ac+bc$. 5. $abc\,(a+b+c)$. 9. $a\,(a-b-c)$.
10. $2b\,(b+c-d)$. 11. $ab\,(a-b-c)$. 12. $xyz\,(x+y-z)$.

Exercise III (b). p. 44.

1. x^2+6x+9. 2. $y^2+12yz+36z^2$. 3. $1+6a+9a^2$.
4. $a^2b^2+2a^2bc+a^2c^2$. 5. $a^4+2a^3b+a^2b^2$. 6. $100p^2+60pr+9r$
7. (*a*) $24xy$; (*b*) $210ab$; (*c*) $2ab^2c$; (*d*) $64b^2$; (*e*) $100a$; (*f*) $64y$
(*g*) $\frac{b^2}{4}$; (*h*) $8b^3$.
8. (*a*) 169; (*b*) 144; (*c*) 125; (*d*) $x^2+16x+64$.
9. 6.

Exercise III (c). p. 46.

1. x^2-6x+9. 2. $y^2-16y+64$.
3. $9a^2-24ab+16b^2$. 4. $144a^2b^2+1-24ab$.

5. $81a^2-126a+49$. 6. $x^2y^2-2x^2yz+x^2z^2$.
7. $120ab$. 8. $144x^2yz$. 9. $4x^2y^2$.
10. $4a^2bc$. 11. $120a$. 12. $4y^2$.
13. $25b^2c^2$. 14. $9z^2$. 15. $\frac{a^2}{4}$.
16. $\frac{9}{4}$. 17. $\frac{b^2}{4a^2}$. 18. (*a*) 64; (*b*) 361; (*c*) p^2-2q+q^2.
19. 10. 21. 998004, 9409. 485809 99980001.

Exercise III (d). p. 48.

1. (*a*) 96, (*b*) 55, (*c*) 396, (*d*) x^2-1.
2. (*a*) x^2-a^2, (*b*) p^2-q^2, (*c*) $16a^2-9y^2$, (*d*) $144x^2-25y^2$.
3. (*a*) $(8+x)(8-x)$; (*b*) $(x+2)(x-2)$; (*c*) $(x+10)(x-10)$, (*d*) $(x+50)(x-50)$; (*e*) $(a+b+c)(a+b-c)$. (*f*) $(7x+9y)(7x-9y)$; (*g*) $(x+4)(x-2)$, (*h*) $(3y+3)(1-y)$.
4. (*a*) 720, (*b*) 396, (*c*) 12000, (*d*) 159856, (*e*) 979996. (*f*) 3999999, (*g*) 840000, (*h*) 440000.
5. 10.

Exercise III (e). p. 51.

1. $(c+d)(x+y)$. 2. $(x-y)(c-d)$. 3. $(x-y)(c+d)$.
4. $(c-d)(x+y)$. 5. $(2a+3b)(4c-3d)$. 6. $(x+1)(a+1)$.
7. $(x-1)(a-b)$. 8. $(a+b)(a+c)$.

Exercise III (f). p. 52.

1. x^2+8x+7. 2. $x^2+10x+16$. 3. $9x^2+24x+15$.
4. $x^2-4x-32$. 5. $4x^2+12x-7$. 6. $9x^2+6x-35$.
7. $49a^2+28ab+3b^2$. 8. $9a^2-9a-10$. 9. 567.
10. 3528. 11. 10197. 12. 8272.

Exercise III (g). p. 54.

1. $(x-2)(x-5)$.
2. $(x-1)(x-10)$.
3. $(x-4)(x-5)$.
4. $(x-10)(x+9)$.
5. $(x-5)(x+3)$.
6. $(3x+3)(3x+5)$.
7. $x=\frac{1}{2}$, *or* $\frac{3}{5}$.
8. $x=-\frac{1}{2}$, *or* $-\frac{1}{3}$.
9. $x=-\frac{1}{3}$ *or* $-\frac{1}{4}$.
10. $y=1$ *or* $-\frac{2}{3}$.

Exercise III (h). p. 55.

1. $bx^2+17x+12$. 2. $18x^2+15x-42$. 3. $27x^2-93x+72$.
4. $90x^2-111x+27$. 5. $56x^2-69x+18$. 6. $32x^2-76xy+45y^2$.

7. 3358. 8. 5762. 9. 9765. 10. 17018.

Exercise III (i). p. 55.

1. $(3x+2)(4x-1)$. 2. $(2x+3)(4x+9)$. 3. $(5x+9)(9x+5)$
4. $(5a-4)(2a+3)$. 5. $x=\frac{2}{3}$ *or* $\frac{1}{7}$—6—$\frac{4}{5}$, $\frac{5}{4}$.

Exercise III (j). p. 57.

1. x^4+x^2+1. 2. a^4-2a^2+1. 3. $a^2+b^2+2ab-c^2$
4. $a^2+b^2-2ab-c^2$. 5. x^3+1. 6. x^3-1. 7. x^3-8.
8. x^3+8. 9. a^4+4. 10. a^3-8.

Exercise III (k). p. 57.

2. $(x+4)(x+6)$; $(x+5)^2-1^2$. 3. (*a*) 231 ; (*b*) ·011 ; (*c*) 62 (*d*) 100·75.
4. L. $= 3a^2-2al-l^2$, B. $= 1$ or L. $= a-b$, B. $= 3a+b$.
5. 317800. 6. 9·67 sq. in.
7. $6507\frac{1}{9}$ sq. yds. 8. (*a*) $\frac{11}{17}$; (*b*) $1\frac{24}{131}$.
9. 120 ft. 10. 3252480 ft.
11. 2928 ft.
13. $(2x+15)(x-2)$, $(2x+15)(x-3)$, $x = -7\frac{1}{2}$.
16. (*a*) 2000002 ; (*b*) 20002. 17. $(88+10)(88-10)$.
18. (*a*) 1003003001 ; (*b*) 1·027243729.
19. (*a*) 997002999 ; (*b*) ·994011992.
20. 61023·378 cub. in.
21. $(x+1)(x+5)$, $165 = 11 \times 15$.
23. (1) 1 ; (2) ·00000111 ; (3) ·001751.
25. When $x = 3$ or 5. 26. $x = 7$ or 8.
27. $x = 1$ or 3. 28. $x = \frac{1}{3}$ or -3.
29. $x = 3\frac{1}{3}$ or $\frac{2}{3}$. 31. 2·345 cub. in.
32. 4·869 cub. in.

Exercise IV (a). p. 65.

1. 30 ft. 6 in. 2. Rs. 257-5-9

Exercise IV (b) p. 68.

1. 18 plots. 2. 120 bags.

Exercise IV (c) p. 72.

1. Rs. 60. 2. Rs 1134-8-3. 3. Rs. 7-3-6.
4. Rs. 275 4-0. 5. 18·15 acres. 6. Rs. 33-12-0.
7. 40 yds. 8. Rs. 15-5 as. 9. Rs. 17-8-0.

10. Rs. 1312-8-0. 11. 45 pumps. 12. 216 men.
13. Rs. 112. 14. Rs. 158-2-0. 15. 6 oz.
16. 55 additional men must be employed. 17. 1·568 lb.
18. 75 miles per hour. 19. $21\frac{3}{7}$ minutes.
20. $3\frac{1}{3}$ hrs.

Exercise IV (d). p. 78.

1. Rs. 430-13-4. 2. 114 men. 3. Rs. 8, 78, 850.
4. 79. 5. Rs. 15-12-9. 6. Rs. 114720-11-11.
7. 253·125 sq. in. 8. 8 in. 9. 150·18 cm.
10. 324 lbs. 11. $17\frac{7}{9}$ gallons per minute.
12. 3·6 ; 14·4 ; 32·4. 13. $a = 6$, $b = 5$. 14. 144 : 9.
15. area = 7·625 × (circumference)^2. 16. $t = 67d = \frac{2}{3}t^2$.
17. 39·351 kg. less. 18. $2313\frac{13}{27}$ lbs. 19. 341·25 knots.
20. 2·02 nearly. 21. 1·69 times ; read 1·3 instead of 2·5.
23. 1000 : 864 : 675. 24. 13·6″. 25. D : d :: 28 : 3.
26. Rs. 310. 27. $\frac{1}{8}$ a maund—4 as. per maund.
28. $t = \left(8 + \frac{w-48}{6}\right)$ $\Big\}$ $\frac{w-48}{6}$ should be with an integer or made next higher integer in case there is a remainder.
$t = 16 + 2\left(\frac{w-48}{6}\right)$. $\Big\}$
29. Rs. 4-14 as. 30. $W = \frac{1}{352} d D^2$. where d and D denote the depth and diameter.
31. 1·20. 32. Rs. 120. 33. 11·8 yrs.
34. h varies as t^2. 35. P varies as Z, 2562·4, 15·6 ft.
39. $a = 1$, $b = ·001205$. 40. 120·2.
41. 355·5$\frac{7}{8}$lbs. 42. Is proportional.
45. $a = ·135$, $b = ·09$. 46. Rs. 660.
47. 2784, $s = 40t + 16t^2$. 48. $6\frac{1}{4}$, $4 = \frac{x}{2} + \frac{3}{x}$.

Exercise V (a). p. 89.

1. $x=5$. 2. $x=10$. 3. $x=\frac{17}{27}$. 4. $x=2$.
5. $x=6\frac{4}{7}$. 6. $x=-9\frac{4}{5}$. 7. $x=-1$. 8. $x=-1\frac{2}{9}$.
9. $x=-\frac{7}{17}$. 10. $x=31\frac{1}{2}$. 11. $x=\frac{-13}{27}$. 12. $x=\frac{2}{3}$.
13. $x=0$. 14. $x=\frac{27}{122}$. 15. $x=21\frac{3}{11}$. 16. $x=\frac{9}{94}$.
17. $x=22\frac{7}{19}$. 18. $x=\frac{166}{191}$. 19. $\frac{43}{57}\frac{1}{7}$. 20. $x=-73\frac{1}{3}$.
21. $x=\frac{45}{31}$. 22. $x=\frac{76}{1987}$. 23. $x=\frac{9}{7}$. 24. $x=10$.

25. $x=1$. 26. $x=3\frac{9}{11}$. 27. $x=1\frac{4}{7}$. 28. $x=9\frac{6}{7}$.

29. $x=-\frac{3}{8}$. 30. $x=8{\cdot}2$. 31. $x=\frac{-2}{15}$. 32. $x=\frac{26}{69}$.

33. $x=\frac{-2740}{10647}$. 35. $x=\frac{17}{43}$.

Exercise V (b). p. 92.

1. $ab=4(c-d)$. 2. $\frac{p}{q}=r+s+5$.

3. $m+3=2(s+3)$. 4. $z=100+x+xy$.

5. $ah-l=bk$ or $ah-bk=l$. 6. $d=\sqrt[3]{\frac{280a}{b}}$.

7. $a^3+b^3+c^3=8d^3$. 8. $z^2+y^2=z^2$.

9. $\frac{1}{2}bh=s^2$. 10. $a+b-\sqrt{a^2+b^2}=4$.

11. $\frac{px}{x+y+z}$, $\frac{py}{x+y+z}$ and $\frac{pz}{x+y+z}$. 12. $A=\frac{1}{2}d_1d_2$.

13. $\pi m^2-\pi b^2=s$. 14. $bm-by=xy$.

15. $h(l+b)=h'(l'+b')$. 16. $110\sqrt{10s}$ yds.

17. $\left(a-\frac{d}{b}\right)\left(b-\frac{d}{b}\right)\left(c-\frac{d}{b}\right)=\frac{h}{1000}$.

18. $\frac{px}{x+y+z}=\frac{py}{x+y+z}+\frac{pz}{x+y+z}$. 19. $q-\frac{1}{q}=q^2$.

Exercise V (c). p. 104.

1. 36. 2. 5. 3. 220. 4. 4, 5 and 6.

3. A = Rs 28, B = Rs. 56-4-0, C = Rs. 3-12-0.

6. B = Rs. 693-5-4, A = 980-0-0, C = 1506-10-8.

7. 144 two-anna pieces. 8. 36 in. 9. 50 oranges.

10. Rs. 2. 11. Rs. 1200. 12. 15 days.

13. Rs. 400. 14. £1160. 15. Rs. 715.

16. 5 years. 17. 95. 18. 18 maunds.

19. $17\frac{1}{2}$ miles. 20. $16\frac{37}{48}$ miles from P, 6 hrs. $42\frac{1}{2}$ m. after. A's stars. 21. In 25 hours, $31\frac{1}{2}$ miles from P.

22. 12 miles per hour. 23. At 11-40 at 30 miles from Mysore.

24. $7\frac{13}{21}$ miles. 25. $56\frac{34}{51}$ miles per hour.

26. $32\frac{8}{11}$ minutes past 2, $54\frac{6}{11}$ minutes past 2.

27. $5\frac{5}{11}'$ past 7 : $21\frac{9}{11}'$ past 7 and $54\frac{6}{11}'$ past 7.

28. $13\frac{11}{13}'$ past 3 ; $41\frac{7}{13}'$ past 3. 29. 84·48 yds. nearly.

30. $21\frac{9}{11}'$ past 4, or 1 hr. $5\frac{5}{11}'$. 31. $2\frac{1}{2}$, 5 nearly.

32. 2100. 33. 41 horses. 34. $22\frac{2}{9}$ yds.
35. PA=3·2 cm. and PB=2·4 cm.; PA=22·4 cm.; PB=16·8 cm.
36. 110 miles. 37. 6·3 seconds. 38. 27 miles.
39. $6\frac{1}{4}$". 40. 192 yds. 41. 50 and 51.
42. very nearly 152 grams. 43. 20 grams.
44. $4\frac{3}{16}$ inches from the one inch ball. 45. 48 grams.

Exercise VI (a). p. 111.

1. 358·35 and 358·45 sq. yds. 2. 93·2865 and 93·2875 c.c.
3. £4·0375 and 4·0385. 4. ·176 of a yd.

Exercise VI (b). p. 116.

1. (*a*) 2nd, ±·002; (*b*) 2nd ±·002.
2. (*a*) 3rd, 0·001; (*b*) 3rd ·0001.
3. (*a*) ±·2; (*b*) ±·2. 4. 0·333333. 5. 5000.
6. 1·7183. 7. 0·1996. 8. 0·44.

Exercise VI (c). p. 123.

1. (*a*) 3·5672; (*b*) 4·7726: (*c*) 8·8072; (*d*) 2·5839; (*e*) ·76921 (*f*) 640. 2. 1. 3. 25600. 4. 1·00172. 5. 1·2.
6. (*a*) (i) to the units' place; with 1 in the 1st decimal place.
(ii) to 3 places with 2 in the 4th decimal place.
(iii) to 5 places with 2 in the 6th decimal place.
(*b*) (i) to 1 place with 2 in the 2nd decimal place.
(ii) to 3 places with 1 in the 4th decimal place.
(iii) to 4 places with 2 in the 5th decimal place.
7. 34500 ft. 8. 8·727: 8·16: 12·34: 11·99.
9. Exports increase 11·17 %. Imports decrease 29·57 %.
10. 33·3 acres. 11. 20·2. 12. 2184 sovereigns.
13. 428. 14. 25 %. 15. 23,700.

Exercise VI (d). p. 129.

1. 21·70. 2. 1482·77. 3. ·25.
4. 3723. 5. 54016. 6. 54600000.
7. 33500000. 8. ·0000476. 9. ·0460.
10. 3,548,000,000. 11. $a \times (10)^{10}$. 12. 340 in.
13. 376. 14. 73 km. per hour.
15. Rs. 2027-5-10.

Exercise VI (e). p. 134.

1. 1·317. 2. ·1508. 3. 20·071.

4. ·0871. 5. 24·524. 6. ·028
7. ·002. 8. ·008. 9. ·0544.
10. ·3183. 11. 1·6348. 12. 4·2940.
13. ·09104. 14. ·3314. 15. 587·3.
16. 20·5. 17. 1 m. = 39·38 in.
18. 1 in. = ·02539 m. *i.e.*, 25·4 mm. 19. 12·4.
20. 13940000.

Exercise VI (f). p. 136.

1. (*a*) 1·713, (*b*) 1550·0048. 2. Rs. 8602.
3. 246. 4. ·3183, 32 inches.
5. ·009211 ; 12711 km. 6. 483 millions.
7. 334. 8. 277.
9. 588×10^{25} grammes. 10. 2·3.
11. Rs. 2·1. 12. 10546233 gr. per sq. cm.
13. ·31. 14. 15·1 cm.
15. 2986 mm. 16. 2·6 pies per m., third, 10, 4 pies, 20, 8 pies.
17. 2·1 pies per lb.
18. (1) 1200, (2) 152, (3) ·0004, (4) 22·3, (5) 1×10^4, (6) 6784.
19. (i) Rs. 6-13 as. Rs. 5-12 as ; (ii) Rs. 6-5 as.
20. (1) + 2 %, (2) 3 pies. (3) Rs. 4330000.
21. Rs. 10-4-3 ; 13 as. 6 p. 22. Rs. 15-1-4.
23. 160·9307. 24. 24855.
25. 8·778088. 26. 40.
27. 2·471 ; 405. 28. 2·3 lbs.
29. 11·9 % decrease. 30. —76·22.
31. (1) 2804, (2) 1932. 32. 5*s*.-5*d*.

Exercise VI (g) p. 143.

1. ·0003... ; ·03 % approximately. 2. ·0005 roughly.
3. ·0004 roughly.
5. To the first two (1) to three—·94 °/₀ ·07 °/₀.
6. ·000022 roughly (2) ·000023 roughly.
7. to four figures. 10. 1 mm. in excess. 11. 1200 m.
12. 39·4 ; 0·08. 14. 2·2.
15. lat. 45°, Paris, Edinburgh, Greenwich, 0·28°/₀; 0·37°/₀; 0·53°/₀ ; 0·56°/₀ ; 0·59°/₀ ; 0·63°/₀ ; 0·08°/₀.

16. 3 significant figures, 7, 5, ; 31428·6; 31415·9; 31416; 31415·9 each correct so far as it is expressed.

17. 5275 and 5265 tolas; **19.** (*a*) foot. (*b*) a sq. mile (*c*) a lakh; (*d*) a mm.; (*e*) tenth of a degree; a second.

20. 4 figures Rs. 51779000 and 55115000; **21.** 3708 sq. m.; 4·5 sq. m.

25. 0·1 and 0·01 in the case of a pendulum beating seconds.

Exercise VI (h). p. 149.

1. 12194 ± 5 grains. **2.** 160200000 ± 144000.

3. 32·253 ± ·005. **4.** 2983087410 with a possible error in the fourth. **5.** 10·27±1·4 lbs. almost equal.

6. 9410 ± 86 with a possible error in the third.

7. (1) 1·6 °/₀, 32°/₀, 48°/₀.

8. (*a*) (1) ·000057 ; ·000074 (*b*) ·0000041 ; ·00060 (*c*) 0000000077. ·0011, ·00011, ·000011. (2) (*a*) 8, (*b*) ·00006, (*c*) ·02 ; 000002 ; ·0002.

9. (*a*) First 5 significant figures. (*b*) first 6 significant figures.

10. 24850 miles —518 miles per hour nearly.

Exercise VI (i) p. 154.

1. £16-6-11. **2.** Rs. 29-11-9. **3.** Rs. 0-4-8.

4. Rs. 8-15-7. **5.** £15-0-4. **6.** Rs. 21-7-6.

7. £0-17-7. **8.** Rs 11-10-10. **9.** Rs. 45-0-0.

10. Rs. 63-1-7. **11.** Rs. 27·349. **12.** Rs. 10·604.

13. Rs. 5·995. **14.** £17·57. **15.** Re. 0·427.

16. £8·479. **17.** Rs. 35·417. **18.** Rs. 9·609.

19. Rs. 16·464 **20.** £19·996.

21. £83·115 ; £83-2-4. **22.** Rs. 2-7-4.

23. Rs. 16615.

Exercise VI (j) p. 154.

4. 93000000; error 400,000.

5. 6·7 yds. nearly to the first two significant figures.

6. 277±1·650 with a possible error in the 'units' place.

8. 3·74±·02 with a possible error in the second decimal place.

9. 27·6±·1 with a possible error in the first decimal place.

10. Rs. 46·9±·9 with a possible error in the first decimal place

11. 1·19±·02. **12.** 94% very nearly. **13.** ·0000003.

15. ·0008 ; ·000007 ; ·00002 ; ·000001 ; 4·129 ; 454·4545... yrs. ; 128·20 yrs. ; 3333·3 yrs. **16.** 65:21. **17.** 13·490±·001.

18. 38·7% ± ·4%. 19. 539·2 cm. ±40·2.
20. 1081·1 cub. in.; 1·0076 : 1; 1·0114 : 1. 21. 988±32.
22. 93300±500. 23. 1·71827515. 24. 8207 cub. m.

Revision Papers—I Series. p. 158.

1.

1. 0. 2. $a(x+y)^2b(x+2y)(x+2y+3)$. 3. 1 sq. ft.
4. 180. 5. (1) ab; (2) $2(a+b)(a-b)$; 1960; 182; 21,
6. $x(a+b)$ sq. in.; 45·5 sq. in. 7. $x=\frac{cd-ab}{a+b-c-d}$.
8. 120'; 1 hour.

2.

1. $x^4-x^2y^2+2xy^3-y^4$. 2. Rs. 698 14 as. 3 pies.
3. Rs. 698 14 as. 3 pies. 4. $43\frac{5}{9}°$ R, $54\frac{4}{9}°$F.
5. 10·7642 sq. ft. 6. $15\frac{20}{57}$. 7. B=Rs. 60, A=Rs. 90.

3.

2. 20120103. 3. 28°. 4. $b=·53$, $a=·22$.
6. ·2743. 7. $10\frac{3}{8}$ in. × $8\frac{2}{3}$. 8. 20 yds. at 7 as. 6 pies and 50 yds. of the other sort.

4.

2. ·22 of a gram. 3. 31 acres 2024 yds.
4. (a) $7y(x+2y)(x-2y)$; (b) $(a+3b+5)(a+3b-5)$.
5. Rs. 5062½. 6. $45\frac{5}{24}$ ft. 7. $x=1\frac{3}{3}\frac{1}{8}\frac{9}{0}$. 8. 88 men.

5.

2. 11111116422061744. 3. 17 millions.
4. Glass platinum = ·0000255, Cast iron = ·00003, Copper = ·000051, Lead = ·000084, Zinc = ·0009.
5. 114622 gallons. 6. From Rs. 1445, 1446 etc.
7. $\frac{5}{12}$ litre water and $\frac{7}{12}$ the other liquid. Rs. 1458.
8. $x=\frac{3}{5}$.

6.

1. ·000600270. 2. 11·1 seconds. 3. 7·8 miles per second.
4. $\frac{86400}{86400+x}\times(60a+b)$, $\frac{86400}{86400-x}\times(60a+b)$ 5' 650 yds.
6. (a) $(x-5y)(x+3y)$; (b) $2(5-2x)(25+10x+4x^2)\{(a+b)-2(a-b)\}\{(a+b)-5(a-b)\}$. 7. 576 sq. ft. 8. $6\frac{12}{13}$.

7.

1. $4a^4+49a^2b^2-28a^3b-25b^4$. 2. ·0000000009 sq. ft. about the same value. 3. 257·0 yds. 4. 125 km. ·005 of an inch.
5. 30·72 miles per hour. 6. (1) 49; (2) 1·80.
7. 2150 copies. 8. Rs. 556 nearly.

8.

1. 3 significant figures. 2. ·0001. 3. 4·15.
4. $26\frac{1}{2}$ miles per hour; 1 hr. 0 m. 18 seconds. 5. 4 : 1.
6. ·228 of an acre. 7. $x=-(a+b+c)$. 8. $3\frac{1}{3}$ miles per hour.

9.

1. 9·8696; 31·0062; ·1014; ·0323. 2. 1·0002 of a ton.
3. Rs. 26 2 annas. 4. Rs. 183-12-0; 93-12-0; 33-12-0.
5. ·0009. 6. earlier by $3\frac{1}{2}$ minutes; $\frac{7}{15}$ mile beyond the usual spot. 7. 11996320. 8. 15·2.

10.

2. x^2-2x+3. 3. ·991; ·002997 %. 4. 39·38.
5. at $1-17\frac{1}{17}$ minutes, $42\frac{6}{17}$; miles from B.
6. (1) $2a(1+3a)(1-3a)$; $(3a-2b)(2a+b)$; (3) $(a+b)^2(a-b)^2$.
7. 2 : 1. 8. $x=-\frac{1}{5}$.

Exercise VII (a). p. 171.

1. (1) 8 rt. ∠s; (2) 12 rt. ∠s; (3) 16 rt. ∠s; (4) 20 rt. ∠s.
2. $\frac{4}{5}$ rt. ∠; (2) $1\frac{1}{2}$ rt. ∠; (3) $1\frac{3}{5}$ rt. ∠; (4) $1\frac{2}{3}$ rt. ∠s.
7. 8 sides. 8. 15 sides.

Exercise VII (b). p. 173.

1. (1) 72°; (2) 60°; (3) 45°; (4) 36°.
2. (1) 108; (2) 120°; (3) 135°; (4) 144°.
3. 6 sides. 4. 36 sides, 72 sides, 360 sides, 144 sides.
5. No, No, No, Yes, Yes, Yes. 6. Yes, Yes, No, Yes, Yes.

Exercise VII (c). p. 178.

1. 120° each. 2. 36°, 72°, 72°. 3. 45°.
4. (1) 90°, 90°, 90°. 5. 72°, 108°. 6. 120°, 135°, $\frac{180^\circ}{n}(n-2)$.

Exercise VII (f). p. 180.

1. 120°. 2. 120°. 4. 40°.

Exercise VII (i). p. 189.

7. 65°, 115°, 115°.

Exercise VII (l). p. 193.

2. 95°. 3. 55°, 105°. 4. B = 36°, A = 108°, C=144°, D=72°. 5. 171°. 6. 36°, 72°, 72°. 7. each ∠=120. 8. 4 rt. angles. 14. Arc of a circle, one Δ, 60°, 60°, 60°. 24. 120°. 25. 18 ft. 26. 15 ft. 27. 2 solutions. 32. 50 ft. 33. 24 ft. by 16 ft. 34. 30·6. 35. 7938 miles; $11\frac{1}{2}$ miles. 36. 40°, 80°, 60°. 44. 4 solutions. 45. 2·7 remaining. 46. A st. lines 111 to the parallel lines midway between them. 47. Bisectors of the angles between the lines. 48. 4 circles, 2 solutions. 49. 1·94 cm. 50. 4·51 cm.; 5·80, 8·13 cm.

Exercise VIII (a). p. 199.

1. 18. 2. 19. 3. 24. 4. 36. 5. 42. 6. 48.

Exercise.—Oral. p. 200.

1. (*a*) 2; (*b*) 3; (*c*) 5; (*d*) 8; (*e*) 9.
2. (*a*) 2; (*b*) 2; (*c*) 3; (*d*) 4 digits.

Exercise.—Oral. p. 205.

(*a*) (1) 2; (2) 8 places; (3) 10 places.
(*b*) (1) 2 and 2; (2) 2 and 6; (3) 6 places and 8 places.
(*c*) (1) 1; (2) 3; (3) 4. (*d*) (1) 1 and 1; (2) 1 and 2; (3) 2 and 1.

Exercise VIII (b). p. 206.

1. (*a*) 46 ft.; (*b*) 65 cm.; (*c*) 95 yds.
2. (*a*) 91; (*b*) 97; (*c*) 105; (*d*) 115; (*e*) 117.
3. (*a*) 139; (*b*) 832; (*c*) 2403; (*d*) 3163; (*e*) 12485; (*f*) 4151. (*g*) 5198; (*h*) 19876; (*i*) 59766 and there is a remainder of $874\frac{4}{5}$. (*j*) 62573. 4. (*a*) $\frac{29}{34}$; (*b*) $\frac{48}{59}$; (*c*) $\frac{139}{1126}$; (*d*) $20\frac{1}{4}$; (*e*) $73\frac{9}{17}$;
5. (*a*) ·32; (*b*) 3·47; (*c*) ·065; (*d*) ·0316; (*e*) ·4927; (*f*) ·4698; (*g*) 10·043; (*h*) ·0359; (*i*) ·15367.

Exercise VIII (c). p. 208.

1. ·7905. 2. ·3118. 3. 2·5298. 4. 27·6568.
2. 2160·008 ft. 3. (*a*) 20·07 (to 2 places); (*b*) 30·03 (to two places; (*c*) 100·000045 (to six places). 4. (1) $(7\frac{7}{8})$; (2) $26_{\frac{1}{13}}$

(3) $32\frac{1}{24}$; (4) $38\frac{77}{78}$; (5) $51\frac{15}{16}$; (6) $73\frac{1}{74}$; (7) $114\frac{21}{26}$: (8) $37\frac{8}{10}$. The fractions should be given in decimals.

Exercise VIII (d). p. 210.

1. 1·4142136. 2. 2·2360680. 3. 2·6457513.
4. 3·3166248. 5. 4·7958615.

Exercise VIII (e). p. 212.

1. 28·12311. 2. 3·46410. 3. ·86603. 4. 3·16228.
5. 8·3588. 6. ·585786. 7. 3·732051. 8. 2·993327.
9. ·236038. 10. 3·025715.

Exercise VIII (f). p. 217.

1. 1320 ft. 2. 57. 3. 34 classes. 4. $68\frac{1}{3}$ ft.
5. 25 ft. 6. 36 minutes. 7. 32 ft.; 32·24 ft. 8. 58 ft.
9. 13 ft. 10. 762 yds. 1 ft. 11. $x = 4, y = 3$.
13. 7·7, 2·72, 8·60. 14. 27·38, 26·92.
15. (1) 2·828; (2) 8·944...; (3) 28·284; (4) 89·443; (5) ·894; (6) ·283; (7) ·0894; (8) 8000: (9) 8944; (10) ·0283.
18. 3·4 ft. 19. 84 men; 25 men. 20. (1) 1 ft. 9 in.; (2) 2 ft.; (3) 2 ft. 3 in. 22. (*a*) 2·7639320; (*b*) 2·7639320.
23. (*a*) ·12 of an inch; (*b*) ·20 of an inch. 24. (1) 810·4; (2) 385·1. 25. 27·7; 138·2. 26. 121·9. 27. 336.
28. (1) 4·24; (2) 3·162; (3) 4·58. 29. 10 acres.
30. *c*C = 2 ft. 3 inches, D*d* = 2 ft. 10 inches, E*e* = D*d* and F*f* = *c*C. 31. ·82. 32. 2·61 in.; ·98 in.
33. 111·73182. 35. 242·72 ft.

Exercise IX (a). p. 226.

1. £771-15*s*. 2. £4466-6-2. 3. Rs. 5534-10-6.
4. Rs. 2857-2-6. 5. £161-0-6. 6. Rs. 215-0-2.
7. £44-8-4. 8. Rs. 120-14-2. 9. Rs. 4239-7-8.
10. £20-7-7. 11. Rs. 1219-7-6. 12. £24-6-0.
13. Rs. 165-12-6. 14. Rs. 2501. 15. £800.
16. £6250. 17. £6250. 18. £500.
19. £709-3-0. 20. 4 p. c. 21. £1-9-6·9*d*.
22. £681-17-8·6*d*. 23. 6*s*. 7·9*d*. 24. £2-7-4.
25. 4*d*. very nearly.

Exercise IX (b). p. 233.

1. (*a*) 1·061414625; (*b*) 1·050625. 2. 2 yrs. 3. 3 yrs.

4. Rs. 15732. 5. 2311414; 1645219. 6. (*a*) £409·76.
(*b*) 28 years. 7. 2076400. 8. 1558151.
9. Rs. 16402·5. 10. 155 : 37. 11. $\frac{1}{64}$ ABCD.
12. $\frac{1}{1024}$ of the original ⊙. 13. ·389 ft. nearly.
14. In 8 years. 15. Rs. 900-2-0 nearly.
16. $r = 237\,^{\circ}/_{\circ}$, Rs. 13·55. 17. 6 years and Rs. 62·155...
18. (*a*) 2; (*b*) 1.

Exercise X (a). p. 239.

1. $x = 2$, $y = 5$. 2. $x = 0$, $y = 0$. 3. $x = 1$, $y = 2$.
4. $x = 3$, $y = 4$. 5. $x = 5$, $y = 7$. 6. $x = 9\frac{2}{3}$, $y = -9\frac{2}{3}$.

Exercise X (b). p. 245.

1. $x = 9, y = 1$. 2. $y = 1, x = 2\frac{1}{2}$.
3. $y = 2, x = 1$. 4. $y = 1, x = 1\frac{1}{2}$.
5. $x = 1, y = 2$. 6. $x = 13, y = 12$.
7. $x = y = -5$. 8. $x = y = 6$.
9. $x = 3, y = 4$. 10. $y = \frac{11}{29}, x = \frac{1}{92}$.
11. $x = -\frac{1132}{13}, y = \frac{-1267}{169}$. 12. $y = 21, x = -85\frac{3}{4}$.
13. $y = -\frac{3}{2}, x = 6\frac{3}{5}$. 14. $x = \frac{437}{290}, y = \frac{-1672}{290}$.
15. $x = 12, y = 0$. 16. $x = \frac{-129}{187}, y = \frac{90}{187}$.
17. $y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$, $x = \frac{b_1c_2 - b_2c_1}{a_2b_1 - a_1b_2}$.
18. $u = \frac{ac + a - b}{4c}$, $v = a$. 19. $x = 0, y = 0$.
20. $x = 0$. $y = 0$.

Exercise X (c). p. 253.

1. B = $\frac{1}{2}$, A = $-\frac{1}{2}$. 2. 4000 and 2239. 3. 50 and 30.
4. ~~36 and 20~~. 5. 1 bottle = $\frac{1}{8}$ gallon; jar = 1 gallon.
6. 1 measure costs Rs. 60-0-0. 7. $8\frac{6}{11}$ as.
8. 2nd = 696 and 1st = 1584. 9. A Rs. 37, B Rs. 17.
10. 1 oz. of silver £0-7-0 and $\frac{2400}{17}$ oz. of gold and $\frac{1000}{17}$ oz. of silver. 11. 1st : 2nd :: 18 : 25. 12. 200 fruits 3 as.

13. Rs. 25. **14.** £96 and 100 quarters. **15.** 1493⅓ lbs.
16. £500, 540. **17.** L = 1193½, B = 163.
18. 10 miles rate and 40 miles distance.
19. Driving 6 miles an hour and walking 3 miles an hour.
20. $\frac{qB - pN}{pq(BM - AN)}$. **21.** 600 and 6 %.
22. 16⅔ and 13⅓ miles per hour. **23.** 44 and 26⅔ yds.
24. 93¾ miles, 22½ and 37½ miles per hour. **25.** 200 miles.
26. 5 min. and 6 min. **27.** 13⅓ min. and 13⅝ min.
28. $m = \frac{1}{2}$, $c = 5$. **29.** $y = x + 1$. **31.** 12, 24, 36, 48.
32. S. G. of the 2nd liquid $= \frac{27}{43}\frac{8}{9}$; of the other liquid $= 5\frac{9}{6}\frac{7}{3}\frac{8}{9}$....
33. 100 and 98. **34.** $a+2b$ rooms and $2a+2b$ travellers.
35. p = 73 + $\frac{9}{23}$ t. **36.** (1) $-17^{\circ}\frac{49}{53}$C; $-15^{\circ}\frac{5}{53}$C; $93\frac{1}{3}^{\circ}$C.
(2) 99°·84F; 28°·8.

Exercise XI (a). p. 259.

1. (*a*) An acute angle; (*b*) An acute angle;
(*c*) An acute angle; (*d*) An acute angle. **2.** $\frac{85}{14}$ or $\frac{85}{20}$.

Exercise XI (b). p. 263.

1. 5118. **2.** 19870. **3.** 1014. **4.** 2481.
5. 5·304, 9·565, 20·026, 6339. **6.** 6 times the area of each in (5).

Exercise XI (c). p. 266.

1. (*a*) 2106; (*b*) 756; (*c*) no triangle can be formed;
(*d*) 1890; (*e*) $\sqrt{\frac{3}{4}}$; (*f*) $\frac{y}{2}\sqrt{x^2 - \frac{y^2}{4}}$.

2. (*a*) 1 : 15; (*b*) 4 : 45; (*c*) 16 : 153.

Exercise XI (d). p. 268.

1. 45·68 ft. **2.** 5130 sq. yds. **3.** 290 yds. **4.** 238 sq. ft

Exercise XI (e). p. 278.

1. 227500 sq. links. **2.** 68800 sq. links.
3. 65937·5 sq. links. **4.** 60825 sq. links.
5. 33175 sq. links. **6.** 60995·2 sq. links.
7. 37906·3 sq. links. **8.** 1130 sq. yds.
9. 1860 sq. yds. **10.** (1) 69⅓ sq. ft.;
(2) 702 sq. ft.; (3) 564 sq. ft.

Exercise XI (g). p. 283.

2. 140 ft., 320 ft. 3. 270 yds., 24300 sq. yds.
4. 160 yds. 5. 500 ft. 7. 88 yds. 8. ·7 of a mile.
9. 310 ft. 10. 30 miles, 4° N. of East.

Exercise XI (h). p. 284.

1. 42 yds. 1 ft. 3 inches. 2. 80 acres. 3. Rs. 135.
4. 1104 sq. ft. 5. 29·568 of an acre.
6. 96 sq. units. 8. 5·1960 sq. in.
9. (1) $\frac{ab}{2}$; (2) $\frac{1}{2}\sqrt{a^2+b^2}$; 30 sq. in. 10. 43·2 sq. inches.
11. 10·39 sq. in. when they are on the same side of the centre. 70·09 sq. in. when on opposite sides. 12. $\frac{65}{7}$ and $\frac{33}{7}$.
13. 13·42 in. 14. 409 sq. chains. 15. 160·99 sq. in.
16. $x\left(\sqrt{w^2-z^2}+\sqrt{y^2-z^2}\right)$. 17. 1·73 sq. in.
18. 32·72 sq. in. 19. 1·72 sq. in. 20. 23·38 sq. in.
21. $\frac{1}{792}$; $2\frac{1}{2}$ miles. 22. $\frac{1}{2534\cdot4}$. 23. 43·75 sq. in.
24. 267 sq. in. 26. 10·5 sq. ft. 27. 775 sq. in.
28. 331 sq. in. 29. 128 sq. ft. 30. ·028 of an acre.
31. 1·89 sq. in. 32. 17·55 sq. cm. 33. Rs. 24-14 as. nearly.
34. 1·288 sq. in, ; Rs. 3-10-10. 35. 2 sq. in. 36. 1·4 in.
40. 1·8 miles an hour. 41. 60 ft. high ; 50·9 yds.
42. 74 ft. (Read 5° instead of 15° and ft. for yd.
43. 93° and 43° at N. and S. respectively.
44. 2050 ft. (Read 55° instead of 45°). 45. 700 ft.

Revision Papers—II Series. p. 289.

1.

1. 3151 full tiles ; 278 △ pieces. 2. 29 ft. ; 72·2 ft.
3. £487 1*s*. 11*d* 5. 20·15 sq. cm. (Read 150° for 30°).
6. 3·46. 7. ·83. 8(*a*). $y=1$, $x=2$; (*b*) 3600, 53°.

2.

1. 2·75 oz. 2. 2A, 90°—A, 90°—A, 360°—2A, A−90°, A−90°.
3. 250230. 4. 11·73 pagodas. 5. 3·36 sq. ft.
6. 33450 sq. chains, 7. 3096. 8. (*a*) ·0627 ; (*b*) $\frac{8}{25}$.

3.

1. (1) $\frac{8}{9}$ of a grains ; (2) $\frac{5}{9}$. 2. $90 - \frac{A}{2}$, $90 - \frac{B}{2}$, $90 - \frac{C}{2}$.
3. 5 ac 13·5 cents. 4. Rs. 2514 12 as. 7 pies.
6. 5·12 ; 6·14 ; 6·88. 7. In 115 seconds.
8(*a*). $g^2 - u^2 = 2fs$; (*b*) 35 : 32·5.

4.

1. $382628\frac{4}{7}$ cub. ft. 2. 2 sq. inches. 3. 2375 ft.
4. £684 14*s*. 5. A = 10, B = 0. 6. 11 and 13 ; 5 : 6 : 7.
7. 32 acres and 6 sq. chains. 8. 28·0622.

5.

1. $90° - \frac{A}{2}$, $90° - \frac{B}{2}$, $90° - \frac{C}{2}$. 2. 2108 in., 1250 inches.
3. Rs. 104166 3 as. 8 ps. 4. $6\frac{3}{7}$ seconds. 5. 27 sq. ft.
7. $79\frac{1}{11}$ yds. 8. 3·6056.

6.

1. $180° - 2A$, $180° - 2B$, $180° - 2C$. 2. 30°.
3. $y = \frac{(3ab + 2a^2 + 5b^2)}{2(3a + b)}$, $x = \frac{3ab - 7b^2 - 4a^2}{3a + b}$.
4. 25 minutes more. 5. Rs. 22 4 as. 11 ps.
6. 17 inches. Cos of $\angle$ opposite to 16 = $\frac{15}{17}$, opposite to 30, $\frac{8}{17}$, opposite to 34 = 0 7. $\frac{1}{16}$ metre, 3 seconds.
8. (*a*) ·9086 ; (*b*) 300 metres.

7.

1. 460·5. 2. Peri. of the Δ : peri. of the S. 1·14 : 1.
3. At the end of the 6th year.
4. $y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$; $x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$. 5. 2 miles.
6. 25 ft. 7. 52 sq. units. 8. 26·9 ft., 875 sq. ft.

8.

2. $\frac{-91}{344}$ and $\frac{-77}{43}$. 3. 36·47. 4. (*a*) 252·495 ; (*b*) 400.
5. 17·5, 62·5 ; 57·5, 2·5 ; 72·5, 36·25 ; 57·5 and 62·5.
6 10 sides and 12 sides, Read 12° as 6°. 7. 2·23 ft.
8. $895\frac{5}{9}$ miles from Madras and $5304\frac{4}{9}$.

9.

1. 5·98; 5 : 9. 2. 6-40 A.M.
3. 1408 sq. inches $391111\frac{1}{9}$ sq. ft. 4. 1·46. 5. $W = \frac{1}{2}mv^2$.
6. 52469·2 sq. units. 7. ·0084 8. $\frac{1}{64}$ △ ABC.

10.

1. 648 sq. ft. 2. 204 ft. 3. 3·46. 4. 1008 sq. cm.
5. $x = 0$, $y = 0$. 6. $y(a - c) = x(b - d) + ad - bc$.
7. 205·5 miles, 205720 ft 8. Between 17 and 18.

Exercise XII (a). p. 302.

1 (i) 9·11; (ii) 23·88; (iii) 43·36. 2. (1) 48·37 :
(2) 31·41; (3) 81·60. 3. (1) 24866 miles;
(2) 24856 miles. 4. $1131\frac{3}{7}$ ft. 5. Rs. 15-15-6.

Exercise XII (b). p. 304.

1. 112·91 inches. 2. $10\frac{2}{3}$ yds. 3. ·2 inches. 4. 13·3 cm.

Exercise XII (c). p. 306.

1. 12·64. 2. ·8584 r^2 where r^2 is the radius of the ⊙.
3. 2827 sq. in. 4. 118 ft. 5. ·80 of an acre.
6. 20 yds. 7. 40 sq. ft. 8. 2·23 : 1.
9. ·022 of a sq. inch. 10. 8·6341 sq. metres.

Exercise XII (d). p. 308.

1. 34 sq. ft. 2. 2 sq. ft. 82 sq. in. 3. $83\frac{1}{3}$ ft.
4. 3 sq. inch and 100 sq. in. 5. 4·2 ft.
6. 3 sq. ft. 8 sq. in.

Exercise XII (e). p. 312.

1. 115·6 cm.; 9079 c.c. 2. 74 sq. cm. 27 c.c.
3. 8 sq. ft. 55 sq. in. 4. 6·50 sq. in 5; about 80 pencils.
5. 29·7 sq. cm. 6. 972π cub. ft.; 378π sq. ft.
7. 107 8 cub. ft. 8. 50 cm. 9. 6·36 m.
10. Its height 28·5; ·89 of an inch radius. 11. 8·55 cub. ft.
12. ·032 sq. in. 13. 1365 sq. cm.; 3885 c.c.
14. $18\frac{1}{2}$ yds. 15. 165·12 sq. in. 16. 37·5 lbs. per sq. inch.
17. Rs. 27. 18. 24·615 oz. 19. 756 tons.
20. 26175 gallons. 21. 174·53 sq. yds. 22. 60·23 sq. ft.

23. 60 lbs. 24. 7·32 lbs. 25. Volumes 32: 171; surfaces 10 : 33. 26. $13\frac{1}{12}$ cub. ft. 27. 5·481 cub. ft.
28. 56·59 ft.; 496084 lbs. 29. 2428 gallons.
30. 900 lbs. (1 cub. ft. of copper weighs 550 lbs.) 31. ·855.
32. 8251 in.; $16\frac{5}{16}\frac{3}{2}$ lbs. 33. 342·9 lbs. 34. 552·6 lbs.
35. 251·8 lbs. 36. 3303·8 lbs. 37. ·29 in. and 1·16 in.

Exercise XIII (a). p. 322.

1. Rs. 397-8-11 ps. 2. £1. 3. £3.
4. Re. 1 14 as. 8 ps. 5. Rs. 31-9-11. 6. Rs. 20-12-9.
7. Rs. 135-14-7.

Exercise XIII (b). p 332.

1. i. (1) Rs. 3350-5-7 ps. ii. (1) Rs. 2179-8-8.
(2) Rs. 5025-8-5 ps. (2) Rs. 3269 5-0.
(3) Rs. 5695-9-6. (3) Rs. 3705-3-6.
iii. (1) Rs. 3822 9 as. 7 ps. iv. (1) Rs. 3747 5 as. 7 p.
(2) Rs. 5733 14 as. 5 ps. (2) Rs. 5621-0-6 p.
(3) Rs. 6498 6 as. 9 ps. (3) Rs. 6370-8 0.
2. (i) £132823 4*s*. 7*d*. (ii) £135677-4-0.
3. £105-2-2. 4. Rs. 247 4 as. 5 p. 5. Rs. 18785 4 as.
6. £1=28·19 francs 15116 francs; and 1 cent.
7. He gains £76-16*s*. 3*d*. nearly, 8. 24·5*d*. per rouble.

Exercise XIV (a). p. 337.

1. 21·94 cub. inches. 2. 138·3908 cub. ft.
3. 84·312 sq. ft.; volume $=40\frac{5}{6}$ cub. ft. 4. 109·6063 cub. ft.
5. 13·81 ft.

Exercise XIV (b). p. 340.

1. 29·144 sq. ft. 2. 68075 sq. cm. 3. 11·33 ft.
4. 506552 sq. cm. 5. (i) 15·91 ft. (ii) 14·7 ft.
6. 1·2 in.; 5·8 in. 7. radius $= 2\frac{2}{3}$ inches, altitude $=12·6$.
8. 263·89 sq. ft. 9. 150·79 c. inches; 39·27 lbs.
10. 4·69 in. 11. 7·3 in. 12. 372·3 cub. ft.
13. 102·44 sq. ft., 166·44 sq. ft. 14. 8·38 inches.

Exercise XIV (c). p. 348.

1. 229·34 sq. inches; 395·4 cub. in. 2. (*a*) 243 sq. cm. 382 cc. (*b*) 308·75 sq. in.; 1047·52 cub. in. (*c*) 6942·94 sq. in.; 135876·7 cub. in. 3. (*a*) $234\frac{2}{3}$ cub. ft.. (*b*) 904·5 cub. in. (*c*) 2218·8 cub. in. 4. 7·14 in. 5. 166^2 cub. in. 6. $319\frac{3}{8}$ c. in.

Exercise XIV (d). p .352.

1. 381·65 cub. in.; 99·3 lbs. 2. 16·9 lbs. nearly.
3. 19·37 cub. in.; 10·60 cub. in. 4. 2·5 inches. 5. 4·04 lb.
6. 201,036,800 sq. miles; $268{\cdot}0406 \times 10^9$ cub. miles.
7. (i) 45233280 sq. miles. (ii) 27921778 sq. miles.
8. Torrid zone = 60,52000 sq. miles; each temperate zone = 40,056000 sq. m.; each frigid zone = 30386400 sq. miles.
9. 2·4 inches. 10. 54·45 cub. in.
11. 530·08 cub. in. 12. 53·87 cub. in.
13. 502·6 sq. inches. 14. 351·18 grams.
15. 77·61 cub. in., neglecting the caps of the spheres.
16. 1380·81 cc. 17. 443·44 sq. in. 18. 75·39 sq. ft.
19. 346·51 sq. ft. nearly; 597⅓ cub. ft. 20. 20·50 sq. in.
21. 2·9 cub. in. 22. Rs. 11-10-5.
23. 2 ft. 7·2 in. nearly; 13740 c. inches. 24. 1122 sq. in.
25. 6 as. 7 pies. 26. Height = 7·2 inches.
27. 4 inches; the product of the radii = 1·82 sq. in.
28. 3·43 nearly. diameter being 5⅔ ft.
29. 320·6 cub. ft. 30. 10944 cub. ft.

Exercise XV. p. 371.

2. £735. 3. £3040 (at 75). 4. £4800.
5. £8130. 6. Rs. 4760. 7. Rs. 3700.
8. £4-6-3. 9. £366-11-3. 10. £705-12-0.
11. Rs. 313-8-0. 12. Rs. 354. 13. £186; £185-15-0.
14. £22-10-0. 15. £63. 16. Rs. 61-4-0.
17. Rs. 76-8-0. 18. £1280. 19. £952-5-0.
20. Rs. 16340. 21. £23,808. 22. £1680.
23. $4\frac{16}{21}$ p.c. 24. $4\frac{1}{6}$ p.c. 25. $4\frac{1}{6}$ p.c.
26. $3\frac{11}{13}$ p.c. 27. £1135. 28. Rs. 6000.
29. £106-13*s*.-4*d*. 30. The 4 percents.
32. 4% and $4\frac{1}{6}$% respectively. 33. £585. 34. 75.
35. $4\frac{1}{6}$. 36. Gains £20. 37. £6710. 38. £456.
39. £5920. 40. It is diminished by £75. 42. £936-10-8.
43. £2641-16-0; £2835 44. $78\frac{2}{9}$, 45. £47 gain.
46. 3·425%. 47. 10,50,000, 2·178 nearly.
48. 800, 1200, 2000, 2000, 2000. 49. Rs. 59299850,
50. (1) Rs. 6000000; (2) Rs. 600; (3) 2,50,000.